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SECOND ORDER ELLIPTIC OPERATORS WITH COMPLEX BOUNDED MEASURABLE COEFFICIENTS IN L^p, SOBOLEV AND HARDY SPACES

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ABSTRACT. – Let L be a second order divergence form elliptic operator with complex bounded measurable coefficients. The operators arising in connection with L, such as the heat semigroup and Riesz transform, are not, in general, of Calderón-Zygmund type and exhibit behavior different from their counterparts built upon the Laplacian. The current paper aims at a thorough description of the properties of such operators in L^p, Sobolev, and some new Hardy spaces naturally associated to L.

First, we show that the known ranges of boundedness in L^p for the heat semigroup and Riesz transform of L, are sharp. In particular, the heat semigroup e^{-tL} need not be bounded in L^p if $p \notin [2n/(n+2), 2n/(n-2)]$. Then we provide a complete description of *all* Sobolev spaces in which L admits a bounded functional calculus, in particular, where e^{-tL} is bounded.

Secondly, we develop a comprehensive theory of Hardy and Lipschitz spaces associated to L, that serves the range of p beyond $[2n/(n+2), 2n/(n-2)]$. It includes, in particular, characterizations by the sharp maximal function and the Riesz transform (for certain ranges of p), as well as the molecular decomposition and duality and interpolation theorems.

RÉSUMÉ. – Soit L un opérateur elliptique du second ordre de formes de divergence, à coefficients complexes bornés et mesurables. Les opérateurs associés à L tels que le semi-groupe de la chaleur ou la transformée de Riesz ne sont en général pas de type Calderón-Zygmund et présentent des comportements différents de leurs analogues construits à partir du laplacien. Cet article a pour objectif de décrire de manière exhaustive les propriétés de ces opérateurs dans L^p, dans les espaces de Sobolev ainsi que dans certains nouveaux espaces de Hardy naturellement associés à L.

Tout d'abord, nous montrons que les plages de valeurs connues pour lesquelles ces opérateurs sont bornés en norme L^p sont strictes. En particulier, le semi-groupe de la chaleur et la transformée de Riesz ne sont pas obligatoirement bornés si $p \notin [2n/(n+2), 2n/(n-2)]$. Nous fournissons ensuite une description complète de *tous* les espaces de Sobolev pour lesquels L admet un calcul fonctionnel borné, en particulier, pour lesquels e^{-tL} est borné.

Puis, nous développons une théorie extensive des espaces de Hardy et de Lipschitz associés à L, pour les valeurs de p hors de $[2n/(n+2), 2n/(n-2)]$. Cette théorie comprend, en particulier, des caractérisations par la fonction maximale « dièse » et par la transformée de Riesz (pour certaines plages de p), ainsi que leur décomposition moléculaire, leur dualité et les théorèmes d'interpolation.

1. Introduction

Let A be an $n \times n$ matrix with entries

$$a_{jk} : \mathbb{R}^n \longrightarrow \mathbb{C}, \quad j = 1, \dots, n, \quad k = 1, \dots, n, \tag{1.1}$$

satisfying the ellipticity condition

$$\lambda |\xi|^2 \leq \Re e A\xi \cdot \bar{\xi} \quad \text{and} \quad |A\xi \cdot \bar{\zeta}| \leq \Lambda |\xi||\zeta|, \quad \forall \, \xi, \zeta \in \mathbb{C}^n, \tag{1.2}$$

for some constants $0 < \lambda \leq \Lambda < \infty$. For such matrices A, our aim in this paper is to present a detailed investigation of Hardy spaces and their duals associated to the second order divergence form operator

$$Lf := -\mathrm{div}(A\nabla f), \tag{1.3}$$

which we interpret in the usual weak sense via a sesquilinear form.

In the case that A is the $n \times n$ identity matrix (i.e., so that L is the usual Laplacian $\Delta := -\mathrm{div}\cdot\nabla$), this theory reduces to the classical Hardy space theory of Stein-Weiss [55] and Fefferman-Stein [32]. For more general operators L whose heat kernel satisfies a pointwise Gaussian upper bound, an adapted Hardy space theory has been introduced by Auscher, Duong and McIntosh [9], and by Duong and Yan, [27], [28]. In the absence of such pointwise kernel bounds, the theory has been developed more recently in [11] by Auscher, McIntosh and Russ (when L is the Hodge-Laplace operator on a manifold with doubling measure), and in [40] by the first two authors of the present paper, for the complex divergence form elliptic operators considered here. In [11, 40], the pointwise Gaussian bounds are replaced by the weaker "Gaffney estimates" (cf. (2.21) and (2.24) below), whose L^2 version is a refined parabolic "Caccioppoli" inequality which may also be proved via integration by parts using only ellipticity and the divergence form structure of L. The present paper may be viewed in part as a sequel to [40], in which we extend results for the case $p = 1$ given there, to the case of general p (although we also obtain here some results, pertaining to the characterization of adapted Hardy spaces via Riesz transforms, that are new even in the case $p = 1$). In particular, it is in the nature of our present setting, in which pointwise kernel bounds may fail, that the Hardy space theory for $p > 1$ becomes non-trivial (i.e., the L-adapted H^p spaces may not coincide with L^p, even when $p > 1$). We shall return to this point momentarily. We note also that general non-negative self-adjoint operators satisfying an L^2 Gaffney estimate have recently been treated in [38].

We now proceed to discuss some relevant history, and to present a more detailed overview of the paper. In [10], the authors solved a long-standing conjecture, known as the Kato problem, by identifying the domain of the square root of L. More precisely, they showed that the domain of $\sqrt{L}$ is the Sobolev space $W^{1,2}(\mathbb{R}^n) = \{f \in L^2 : \nabla f \in L^2\}$ with

$$\|\sqrt{L} f\|_{L^2(\mathbb{R}^n)} \approx \|\nabla f\|_{L^2(\mathbb{R}^n)}. \tag{1.4}$$

In particular, the Riesz transform $\nabla L^{-1/2}$ is bounded in $L^2(\mathbb{R}^n)$.

Since then, substantial progress has been made in the development of the L^p theory of elliptic operators of the type described above. Let us define

$$p_-(L) := \inf\{p : \nabla L^{-1/2} : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n)\}.$$

It is now known that $1 \leq p_-(L) < 2n/(n+2)$ (with $1 < p_-(L)$ for some L; we shall return to the latter point momentarily), and that there exists $\varepsilon(L) > 0$ such that

$$\nabla L^{-1/2} : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n) \quad \Longleftrightarrow \quad p_-(L) < p < 2 + \varepsilon(L), \tag{1.5}$$

(given (1.4) as a starting point, (1.5) with $p_-(L) < 2n/(n+2)$ is established by combining the results and methods of [39] or [18] with those of [6]; see also [5], [13], Chapter 4 of [14], and [17] for related theory). Moreover, again given (1.4) as a starting point, one has the reverse inequality

$$\|\sqrt{L}f\|_{L^p(\mathbb{R}^n)} \lesssim \|\nabla f\|_{L^p(\mathbb{R}^n)}, \qquad \text{for} \qquad (p_-(L))_* < p < (p_-(L^*))', \tag{1.6}$$

where in general $p_* := pn/(p+n)$ denotes the "lower" Sobolev exponent, and as usual $p' := p/(p-1)$ is the exponent dual to p. The case $p < 2$ of (1.6) is due to Auscher [6], while the case $p > 2$ is simply dual to the adjoint version of (1.5). Combining (1.5) and (1.6), we have that

$$\|\sqrt{L}f\|_{L^p(\mathbb{R}^n)} \approx \|\nabla f\|_{L^p(\mathbb{R}^n)} \quad \Longleftrightarrow \quad p_-(L) < p < 2 + \varepsilon. \tag{1.7}$$

One of the main goals of the present paper is to understand the sense in which (1.7) extends to the range $p \leq p_-(L)$. This extension may be viewed as solving the Kato problem below the critical exponent $p_-(L)$. We discuss this question in more detail in Subsection 1.2 below; the proofs are given in Section 5 (cf. Theorem 5.2).

Let us now discuss optimality of the range of p in (1.5) (hence also that in (1.7)), for the entire class of L under consideration. Even in the case of real symmetric coefficients, the upper bound cannot be improved, in general: for each $p > 2$, Kenig(1) has constructed an operator L whose Riesz transform is not bounded in L^p. In addition, the counterexamples in [50], [8], [25] showed that for some elliptic operator L satisfying (1.1)–(1.3) there is a $p \in (1, 2)$ such that the Riesz transform is not bounded in L^p; i.e., for such L, one has $p_-(L) > 1$. Moreover, the latter fact permeates all the L^p results in the theory: as shown in [6], $p_-(L)$ is also the lower bound for the respective intervals of p for which the heat semigroup and the L-adapted square function (cf. (1.10) below) are L^p bounded, and for which the semigroup enjoys $L^p \to L^2$ off diagonal estimates. However, identification of the sharp lower bound $p_-(L)$ remained an open problem (posed, along with related questions, in [6], Conjecture 3.14, and in [4], Problem 1.4, Problem 1.5, Problem 1.13).

In Section 2 of the present paper, we observe that the example constructed by Frehse in [34] may be used to resolve these remaining sharpness issues, i.e., to show that $p_\pm(L) = 2n/(n \mp 2) \pm \varepsilon_\pm(L)$, where $(p_-(L), p_+(L))$ is the interior of the interval of L^p boundedness of the heat semigroup $e^{-tL}, t > 0$. More precisely, we have

$$\forall\, p \notin [2n/(n+2), 2], \quad \exists\, L \text{ with } \quad \nabla L^{-1/2} : L^p(\mathbb{R}^n) \not\longrightarrow L^p(\mathbb{R}^n), \tag{1.8}$$

$$\forall\, p \notin [2n/(n+2), 2n/(n-2)], \quad \exists\, L \text{ with } \quad e^{-tL} : L^p(\mathbb{R}^n) \not\longrightarrow L^p(\mathbb{R}^n). \tag{1.9}$$

It follows, in particular, that in dimensions $n \geq 3$, the kernel of the heat semigroup may fail to satisfy the pointwise Gaussian estimate

$$|K_t(x,y)| \leq C t^{-n/2} e^{-c|x-y|^2/t}, \quad t > 0 \text{ and } x, y \in \mathbb{R}^n.$$

(1) Kenig's Example is described in [14], Section 4.2.2.

This solves an open problem in [14], p. 33.

Thus, in dimensions $n > 2$, the Riesz transform may fail to be bounded in L^p for some $p \in (1,2)$, as may the heat semigroup e^{-tL}, $t > 0$, as well as the other natural operators associated with such L (e.g., square function, non-tangential maximal function). Consequently, in the case that the endpoint $p_-(L) > 1$, the L-adapted Riesz transforms, semigroup and square function cannot be bounded from the classical Hardy space H^1 into L^1, since interpolation with the known L^2 bound would then produce a contradiction with (1.8), (1.9) (or with the analogous statement for the square function). These operators therefore lie beyond the scope of the Calderón-Zygmund theory and exhibit behavior different to their counterparts built upon the Laplacian.

By analogy to the classical theory then, this motivates the introduction of a family of L-adapted Hardy spaces H^p_L for all $0 < p < \infty$, *not* equal to L^p in the range $p \leq p_-(L)$, on which the L-adapted semigroup, Riesz transforms and square function are well behaved, and which comprise a complex interpolation scale including L^p for $p_-(L) < p < p_+(L)$. We note that the endpoint $p_-(L)$ plays a similar role to the exponent $p = 1$ in the classical theory.

In particular, in Section 5 we give a suitable Hardy space extension of (1.5) to the case $p \leq p_-(L)$ (the case $p = 1$ already appeared in [40]), and, in one of the main results of this paper, we present an appropriate converse, thus obtaining a Riesz transform characterization of L-adapted H^p spaces, for some range of p depending on n. As observed above, this characterization may be viewed as a sharp extension of the Kato square root estimate (1.4), and of its L^p version (1.7), to the endpoint $p_-(L)$ and below. In order to make these notions precise, we should first define our adapted H^p_L spaces.

1.1. Definition of H^p_L

The first step in the development of an L-adapted Hardy space theory, in the case that pointwise kernel bounds may fail[(2)], was taken in [40] (and independently in [11]), in which the authors considered the model case of $H^1_L(\mathbb{R}^n)$ and, on the dual side, the appropriate analogue of the space BMO. The definition of H^1_L given in [40][(3)] (by means of an L-adapted square function) can be extended immediately to $0 < p \leq 2$ and with some additional care to $2 \leq p < \infty$ as well. To this end, consider the square function associated with the heat semigroup generated by L

$$(1.10)\qquad Sf(x) = \left(\iint_{\Gamma(x)} |t^2 L e^{-t^2 L} f(y)|^2 \, \frac{dydt}{t^{n+1}} \right)^{1/2}, \qquad x \in \mathbb{R}^n,$$

where, as usual, $\Gamma(x) = \{(y,t) \in \mathbb{R}^n \times (0,\infty) : |x-y| < t\}$ is a non-tangential cone with vertex at $x \in \mathbb{R}^n$. Analogously to [40], we define the space $H^p_L(\mathbb{R}^n)$ for $0 < p \leq 2$ as the completion of $\{f \in L^2(\mathbb{R}^n) : Sf \in L^p(\mathbb{R}^n)\}$ in the norm

$$(1.11)\qquad \|f\|_{H^p_L(\mathbb{R}^n)} := \|Sf\|_{L^p(\mathbb{R}^n)}.$$

(2) In the presence of *pointwise* Gaussian heat kernel bounds, an L-adapted H^1 and BMO theory was previously introduced by Duong and Yan [27], [28].
(3) And in [11] for $H^p_L, p \geq 1$.