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WEAK SYMPLECTIC FILLINGS AND HOLOMORPHIC CURVES

BY KLAUS NIEDERKRÜGER AND CHRIS WENDL

ABSTRACT. – We prove several results on weak symplectic fillings of contact 3-manifolds, including: (1) Every weak filling of any planar contact manifold can be deformed to a blow up of a Stein filling. (2) Contact manifolds that have fully separating planar torsion are not weakly fillable—this gives many new examples of contact manifolds without Giroux torsion that have no weak fillings. (3) Weak fillability is preserved under splicing of contact manifolds along symplectic pre-Lagrangian tori—this gives many new examples of contact manifolds without Giroux torsion that are weakly but not strongly fillable.

We establish the obstructions to weak fillings via two parallel approaches using holomorphic curves. In the first approach, we generalize the original Gromov-Eliashberg "Bishop disk" argument to study the special case of Giroux torsion via a Bishop family of holomorphic annuli with boundary on an "anchored overtwisted annulus". The second approach uses punctured holomorphic curves, and is based on the observation that every weak filling can be deformed in a collar neighborhood so as to induce a stable Hamiltonian structure on the boundary. This also makes it possible to apply the techniques of Symplectic Field Theory, which we demonstrate in a test case by showing that the distinction between weakly and strongly fillable translates into contact homology as the distinction between twisted and untwisted coefficients.

RÉSUMÉ. – On montre plusieurs résultats concernant les remplissages faibles de variétés de contact de dimension 3, notamment : (1) Les remplissages faibles des variétés de contact planaires sont à déformation près des éclatements de remplissages de Stein. (2) Les variétés de contact ayant de la torsion planaire et satisfaisant une certaine condition homologique n'admettent pas de remplissages faibles – de cette manière on obtient des nouveaux exemples de variétés de contact qui ne sont pas faiblement remplissables. (3) La remplissabilité faible est préservée par l'opération de somme connexe le long de tores pré-lagrangiens — ce qui nous donne beaucoup de nouveaux exemples de variétés de contact sans torsion de Giroux qui sont faiblement, mais pas fortement, remplissables.

On établit une obstruction à la remplissabilité faible avec deux approches qui utilisent des courbes holomorphes. La première méthode se base sur l'argument original de Gromov-Eliashberg des « disques de Bishop ». On utilise une famille d'anneaux holomorphes s'appuyant sur un « anneau vrillé ancré » pour étudier le cas spécial de la torsion de Giroux. La deuxième méthode utilise des courbes holomorphes à pointes, et elle se base sur l'observation que, dans un remplissage faible, la structure symplectique peut être déformée au voisinage du bord, en une structure hamiltonienne stable. Cette observation permet aussi d'appliquer les méthodes à la théorie symplectique de champs, et on montre dans un cas simple que la distinction entre les remplissabilités faible et forte se traduit en homologie de contact par une distinction entre coefficients tordus et non tordus.

0. Introduction

The study of symplectic fillings via J-holomorphic curves goes back to the foundational result of Gromov [25] and Eliashberg [9], which states that a closed contact 3-manifold that is overtwisted cannot admit a weak symplectic filling. Let us recall some important definitions: in the following, we always assume that (W, ω) is a symplectic 4-manifold, and (M, ξ) is an oriented 3-manifold with a positive and cooriented contact structure. Whenever a contact form for ξ is mentioned, we assume it is compatible with the given coorientation.

Definition 1. – A contact 3-manifold (M, ξ) embedded in a symplectic 4-manifold (W, ω) is called a *contact hypersurface* if there is a contact form α for ξ such that $d\alpha = \omega|_{TM}$. In the case where $M = \partial W$ and its orientation matches the natural boundary orientation, we say that (W, ω) has *contact type boundary* (M, ξ), and if W is also compact, we call (W, ω) a *strong symplectic filling* of (M, ξ).

Definition 2. – A contact 3-manifold (M, ξ) embedded in a symplectic 4-manifold (W, ω) is called a *weakly contact hypersurface* if $\omega|_\xi > 0$, and in the special case where $M = \partial W$ with the natural boundary orientation, we say that (W, ω) has *weakly contact boundary* (M, ξ). If W is also compact, we call (W, ω) a *weak symplectic filling* of (M, ξ).

It is easy to see that a strong filling is also a weak filling. In general, a strong filling can also be characterized by the existence in a neighborhood of ∂W of a transverse, outward pointing *Liouville vector field*, i.e., a vector field Y such that $\mathcal{L}_Y \omega = \omega$. The latter condition makes it possible to identify a neighborhood of ∂W with a piece of the symplectization of (M, ξ); in particular, one can then enlarge (W, ω) by symplectically attaching to ∂W a cylindrical end.

The Gromov-Eliashberg result was proved using a so-called *Bishop family* of pseudoholomorphic disks: the idea was to show that in any weak filling (W, ω) whose boundary contains an overtwisted disk, a certain *noncompact* 1-parameter family of J-holomorphic disks with boundary on ∂W must exist, but yields a contradiction to Gromov compactness. In [9], Eliashberg also used these techniques to show that all weak fillings of the tight 3-sphere are diffeomorphic to blow-ups of a ball. More recently, the Bishop family argument has been generalized by the first author [36] to define the *plastikstufe*, the first known obstruction to symplectic filling in higher dimensions.

In the meantime, several finer obstructions to symplectic filling in dimension three have been discovered, including some which obstruct strong filling but not weak filling. Eliashberg [12] used some of Gromov's classification results for symplectic 4-manifolds [25] to show that on the 3-torus, the standard contact structure is the only one that is strongly fillable, though Giroux had shown [22] that it has infinitely many distinct weakly fillable contact structures. The first examples of tight contact structures without weak fillings were later constructed by Etnyre and Honda [18], using an obstruction due to Paolo Lisca [30] based on Seiberg-Witten theory.

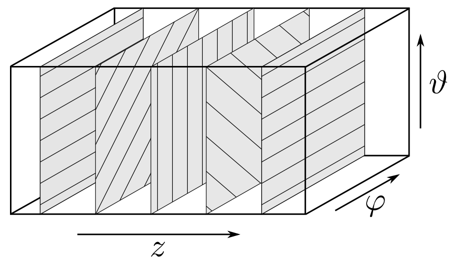

FIGURE 1. The region between the grey planes on either side represents half a Giroux torsion domain. The grey planes are pre-Lagrangian tori with their characteristic foliations, which show the contact structure turning along the z-axis as we move from left to right. Domains with higher Giroux torsion can be constructed by gluing together several half-torsion domains.

The simplest filling obstruction beyond overtwisted disks is the following. Define for each $n \in \mathbb{N}$ the following contact 3-manifolds with boundary:

$$T_n := \left(\mathbb{T}^2 \times [0,n],\ \sin(2\pi z)\, d\varphi + \cos(2\pi z)\, d\vartheta\right) ,$$

where (φ, ϑ) are the coordinates on $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$, and z is the coordinate on $[0,n]$. We will refer to T_n as a *Giroux torsion domain*.

DEFINITION 3. – Let (M,ξ) be a 3-dimensional contact manifold. The *Giroux torsion* $\mathrm{Tor}(M,\xi) \in \mathbb{Z} \cup \{\infty\}$ is the largest number $n \geq 0$ for which we can find a contact embedding of the Giroux torsion domain $T_n \hookrightarrow M$. If this is true for arbitrarily large n, then we define $\mathrm{Tor}(M,\xi) = \infty$.

REMARK. – Due to the classification result of Eliashberg [8], overtwisted contact manifolds have infinite Giroux torsion, and moreover, one can assume in this case that the torsion domain $T_n \subset M$ separates M. It is not known whether a contact manifold with infinite Giroux torsion must be overtwisted in general.

The present paper was motivated partly by the following fairly recent result.

THEOREM (Gay [19] and Ghiggini-Honda [21]). – *A closed contact 3-manifold (M,ξ) with positive Giroux torsion does not have a strong symplectic filling. Moreover, if it contains a Giroux torsion domain T_n that splits M into separate path components, then (M,ξ) does not even admit a* weak *filling.*

The first part of this statement was proved originally by David Gay with a gauge theoretic argument, and the refinement for the separating case follows from a computation of the Ozsváth-Szabó contact invariant due to Paolo Ghiggini and Ko Honda. Observe that due to the remark above on overtwistedness and Giroux torsion, the result implies the Eliashberg-Gromov theorem.

As this brief sampling of history indicates, holomorphic curves have not been one of the favorite tools for defining filling obstructions in recent years. One might argue that this is unfortunate, because holomorphic curve arguments have a tendency to seem more geometrically natural and intuitive than those involving the substantial machinery of Seiberg-Witten theory or Heegaard Floer homology—and in higher dimensions, of course, they are still the only tool available. A recent exception was the paper [42], where the second author used families of holomorphic cylinders to provide a new proof of Gay's result on Giroux torsion and strong fillings. By similar methods, the second author has recently defined a more general obstruction to strong fillings [44], called *planar torsion*, which provides many new examples of contact manifolds (M, ξ) with $\mathrm{Tor}(M, \xi) = 0$ that are nevertheless not strongly fillable. The reason these results apply primarily to *strong* fillings is that they depend on moduli spaces of *punctured* holomorphic curves, which live naturally in the noncompact symplectic manifold obtained by attaching a cylindrical end to a strong filling. By contrast, the Eliashberg-Gromov argument works also for weak fillings because it uses compact holomorphic curves with boundary, which live naturally in a compact almost complex manifold with boundary that is pseudoconvex, but not necessarily convex in the *symplectic* sense. The Bishop family argument however has never been extended for any compact holomorphic curves more general than disks, because these tend to live in moduli spaces of nonpositive virtual dimension.

In this paper, we will demonstrate that both approaches, via compact holomorphic curves with boundary as well as punctured holomorphic curves, can be used to prove much more general results involving *weak* symplectic fillings. As an illustrative example of the compact approach, we shall begin in §1 by presenting a new proof of the above result on Giroux torsion, as a consequence of the following.

THEOREM 1. – *Let (M, ξ) be a closed 3-dimensional contact manifold embedded into a closed symplectic 4-manifold (W, ω) as a weakly contact hypersurface. If (M, ξ) contains a Giroux torsion domain $T_n \subset M$, then the restriction of the symplectic form ω to T_n cannot be exact.*

By a theorem of Eliashberg [14] and Etnyre [16], every weak filling can be capped to produce a closed symplectic 4-manifold. The above statement thus implies a criterion for (M, ξ) to be not weakly fillable—our proof will in fact demonstrate this directly, without any need for the capping result. We will use the fact that every Giroux torsion domain contains an object that we call an *anchored overtwisted annulus*, which we will show serves as a filling obstruction analogous to an overtwisted disk. Note that for a torsion domain $T_n \subset M$, the condition that ω is exact on T_n is equivalent to the vanishing of the integral

$$\int_{\mathbb{T}^2 \times \{c\}} \omega$$

on any slice $\mathbb{T}^2 \times \{c\} \subset T_n$. For a strong filling this is *always* satisfied since ω is exact on the boundary, and it is also always satisfied if T_n separates M.

The proof of Theorem 1 is of some interest in itself for being comparatively low-tech, which is to say that it relies only on technology that was already available as of 1985. As such, it demonstrates new potential for well established techniques, in particular the Gromov-Eliashberg Bishop family argument, which we shall generalize by considering a "Bishop family of holomorphic annuli" with boundaries lying on a 1-parameter family of so-called *half-twisted annuli*. Unlike overtwisted disks, a single overtwisted annulus does not suffice to prove anything: the boundaries of the Bishop annuli must be allowed to vary in a nontrivial family, called an *anchor*, so as to produce a moduli space with positive dimension. One