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STRICHARTZ ESTIMATES FOR WATER WAVES

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ABSTRACT. – In this paper we investigate the dispersive properties of the solutions of the two dimensional water-waves system with surface tension. First we prove Strichartz type estimates with loss of derivatives at the same low level of regularity we were able to construct the solutions in [2]. On the other hand, for smoother initial data, we prove that the solutions enjoy the optimal Strichartz estimates (i.e, without loss of regularity compared to the system linearized at $(\eta = 0, \psi = 0)$).

RÉSUMÉ. – Nous nous intéressons dans cet article aux propriétés dispersives du système des ondes de surface en dimension 2, avec tension de surface. Nous démontrons tout d'abord des estimées de Strichartz, avec pertes de dérivées, au niveau de régularité où nous avons construit des solutions dans [2]. Ensuite, pour des données initiales plus régulières, nous démontrons les estimées de Strichartz optimales (i.e. sans perte de régularité par rapport à celles du système linéarisé en $(\eta = 0, \psi = 0)$).

1. Introduction

In a time-dependent domain $\Omega_t \subset \mathbf{R}^2$ which is located between a free hypersurface Σ_t and a fixed known bottom Γ, consider a potential flow $v = \nabla_{x,y}\phi$, with

$$\Delta_{x,y}\phi = 0 \quad \text{in } \Omega_t, \quad \partial_n \phi = 0 \quad \text{on } \Gamma.$$

The surface-tension water-waves problem is given by two equations: a kinematic condition (which states that the free surface moves with the fluid), and a dynamic condition (that expresses a balance of forces across the free surface). The system reads

$$\begin{cases} \partial_t \eta = \partial_y \phi - \nabla\eta \cdot \nabla\phi & \text{on } \Sigma_t = \{y = \eta(t,x)\}, \\ \partial_t \phi + \dfrac{1}{2}\left|\nabla_{x,y}\phi\right|^2 + g\eta = H(\eta) & \text{on } \Sigma_t, \end{cases} \tag{1.1}$$

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where $\nabla = \partial_x$, $g > 0$ is the acceleration of gravity and

$$H(\eta) = \operatorname{div}\left(\frac{\nabla\eta}{\sqrt{1+(\nabla\eta)^2}}\right)$$

is the mean curvature of the free surface.

1.1. Assumptions

We work in a fluid domain such that there is uniformly a minimum depth of water, more precisely we assume that for each time t one has

$$\Omega_t = \Omega_{1,t} \cap \Omega_2$$

where $\Omega_{1,t}$ is the half space located below the free surface Σ_t,

$$\Omega_{1,t} = \{\, (x,y) \in \mathbf{R} \times \mathbf{R} \,:\, y < \eta(t,x) \,\}$$

for some unknown function η and Ω_2 contains a fixed strip around Σ_t, that means that there exists $h > 0$ such that,

$$\{(x,y) \in \mathbf{R} \times \mathbf{R} \,:\, \eta(t,x) - h \leq y \leq \eta(t,x)\} \subset \Omega_2, \tag{1.2}$$

for all $t \in [0,T]$. We shall also assume that the domain Ω_2 (and hence the domain $\Omega_t = \Omega_{1,t} \cap \Omega_2$) is connected.

We emphasize that no regularity assumption is made on the bottom $\Gamma = \partial\Omega_t \setminus \Sigma_t$. We consider both cases of infinite depth and bounded depth bottoms (and all cases in-between). Finally, we could consider the cases where the free surface is a graph over a given smooth hypersurface and the bottom is time dependent.

1.2. Main results

Following Zakharov we reduce the system to a system on the free surface. If $\psi = \psi(t,x) \in \mathbf{R}$ is defined by

$$\psi(t,x) = \phi(t,x,\eta(t,x)),$$

then $\phi(t,x,y)$ is the unique variational solution of

$$\Delta\phi = 0 \quad \text{in } \Omega_t, \qquad \phi(t,x,\eta(t,x)) = \psi(t,x). \tag{1.3}$$

The Dirichlet-Neumann operator is then defined by

$$(G(\eta)\psi)(t,x) = \sqrt{1+|\nabla\eta|^2}\,\partial_n\phi|_{y=\eta(t,x)} = \partial_y\phi - \nabla\eta\cdot\nabla\phi\,\Big|_{y=\eta(t,x)}$$

(we refer to Section 2 in [2] for a precise construction).

Then (η,ϕ) is solution of the water-waves system (1.1) if and only if (η,ψ) solves the system

$$\left\{\begin{aligned} &\partial_t\eta - G(\eta)\psi = 0,\\ &\partial_t\psi + g\eta - H(\eta) + \frac{1}{2}\,|\partial_x\psi|^2 - \frac{1}{2}\,\frac{\big(\partial_x\eta\cdot\partial_x\psi + G(\eta)\psi\big)^2}{1+|\partial_x\eta|^2} = 0. \end{aligned}\right. \tag{1.4}$$

Concerning the Cauchy theory for the water waves with surface tension, there are many results starting from the pioneering work of K. Beyer and M. Günther [10]. See D. M. Ambrose and N. Masmoudi [6], B. Schweiser [24], T. Iguchi [19], D. Coutand and S. Shkoller

[17], J. Shatah and C. Zeng [25], M. Ming and Z. Zhang [22], F. Rousset and N. Tzvetkov [23]. In [2], we established new local well posedness results for the system (1.4) under sharp (as long as no dispersive effects are taken into account) regularity assumptions on the initial data. We refer to the introduction of [2] for references and a short historical survey of the background of these problems.

The purpose of this work is precisely, in the case $d = 1$, to investigate the dispersive properties of these solutions. Our results are twofold: first we prove Strichartz type estimates with loss of derivatives at the very same level of regularity we were able to construct the solutions in [2]. On the other hand, for smoother initial data, we prove that the solutions enjoy the optimal Strichartz estimates (i.e, without loss of regularity compared to the system linearized at $(\eta = 0, \psi = 0)$).

Define the usual Besov space,

$$u \in B^{\sigma}_{\infty,2}(\mathbf{R}) \iff \sum_{j\in\mathbf{N}} 2^{2j\sigma} \|\Delta_j(u)\|^2_{L^\infty(\mathbf{R})} < +\infty,$$

where $u = \sum_j \Delta_j(u)$ is the standard Littlewood-Paley decomposition of u. Notice that if $\sigma \notin \mathbf{N}$, we have (with continuous injection)

$$B^{\sigma}_{\infty,2}(\mathbf{R}) \subset W^{\sigma,\infty}(\mathbf{R}),$$

where $W^{\sigma,\infty}(\mathbf{R})$ is the usual Hölder C^σ space (which, if $\sigma \notin \mathbb{N}$, is characterized by the fact that $\big(2^{j\sigma}\|\Delta_j(u)\|_{L^\infty(\mathbf{R})}\big)_{j\in\mathbf{N}} \in \ell^\infty(\mathbf{N})$, see for example [15, Proposition 2.3.1]).

Our main results are the following.

THEOREM 1.1. – *Let $s > 5/2$ and $T > 0$. Consider a solution (η,ψ) of (1.4) on the time interval $I = [0,T]$ such that Ω_t satisfies (1.2) for $t \in I$. If*

$$(\eta,\psi) \in C^0\big(I, H^{s+\frac12}(\mathbf{R}) \times H^s(\mathbf{R})\big),$$

then

$$(\eta,\psi) \in L^4\big(I, B^{s+\frac14}_{\infty,2}(\mathbf{R}) \times B^{s-\frac14}_{\infty,2}(\mathbf{R})\big).$$

THEOREM 1.2. – *Let $s > 11/2$, $T > 0$. Consider a solution (η,ψ) of (1.4) on the time interval $I = [0,T]$ such that Ω_t satisfies (1.2) for $t \in I$. If*

$$(\eta,\psi) \in C^0\big(I, H^{s+\frac12}(\mathbf{R}) \times H^s(\mathbf{R})\big),$$

then

$$(\eta,\psi) \in L^4\big(I, B^{s+\frac38}_{\infty,2}(\mathbf{R}) \times B^{s-\frac18}_{\infty,2}(\mathbf{R})\big).$$

REMARK 1.3. – (i) Theorem 1.1 was obtained recently under the assumption $s \geq 15$ by Christianson-Hur-Staffilani [16] .

(ii) Let $s > 5/2$ and $(\eta_0,\psi_0) \in H^{s+\frac12}(\mathbf{R}) \times H^s(\mathbf{R})$ satisfying $\operatorname{dist}(\Sigma_0,\Gamma) \geq c > 0$, we proved in [2] that there exist $T > 0$ and a solution $(\eta,\psi) \in C^0\big([0,T]; H^{s+\frac12}(\mathbf{R}) \times H^s(\mathbf{R})\big)$ satisfying $\operatorname{dist}(\Sigma_t,\Gamma) \geq c > 0$.

(iii) The gain of regularity exhibited in Theorem 1.2 is optimal as can be seen at the level of the linearized system around the trivial solution $(\eta,\psi) = (0,0)$ which reads (when $g = 0$),

$$\partial_t\eta - |D_x|\,\psi = 0, \quad \partial_t\psi - \Delta\eta = 0.$$

Indeed $u = |D_x|^{\frac{1}{2}}\eta + i\psi$ is a solution of the equation $i\partial_t u - |D_x|^{\frac{3}{2}} u = 0$, for which one can prove the optimal estimate

$$\left\|\exp(-it\,|D_x|^{\frac{3}{2}})u_0\right\|_{L^4\left(I,B^{s-\frac{1}{8}}_{\infty,2}(\mathbf{R})\right)} \leq C\|u_0\|_{H^s(\mathbf{R})},$$

which gives the desired regularity on (η,ψ).

(iv) It is most likely that Theorem 1.1 remains valid when $\mathbf{R}$ is replaced by the one dimensional torus $\mathbf{T}$. Indeed, our proof relies on a semi-classical parametrix (on time intervals tailored to the frequency) which exhibits finite speed of propagation and which can consequently be easily localized in space.

(v) Notice that the dispersive estimates proved in this paper can be combined with our previous work to improve the regularity threshold obtained in [2] and give local well posednesss for initial data below the $s = 2 + \frac{1}{2}$ threshold. This will be the matter of a forthcoming paper (including the 3-d water-waves system) [1].

(vi) Notice finally that dispersive properties of the operator linearized at $(\eta = 0,\, \psi = 0)$ were used recently by Wu [30, 31] and Germain-Masmoudi-Shatah [18] to prove global existence results for gravity waves.

1.3. Strategy of the proofs

Following the approach in Alazard-Métivier [3], after suitable paralinearizations, we have shown in [2] that the water waves system can be arranged into an explicit paradifferential symmetric equation of Schrödinger type, and we deduced the smoothing effect for the 2-d surface tension water waves. Here, we will also take benefit of this paralinearization reduction, and this reduced system will be our starting point. The guiding line for the rest of our proof is very classical: construction of a parametrix to prove dispersion ($L^1 - L^\infty$ estimates, and then TT^* argument).

There are two main difficulties in the analysis of this equation. First the coefficients of the operator are time dependent and consequently we cannot get rid of the lower order terms by simple conjugation arguments (see Burq-Planchon [14]). Second the coefficients enjoy poor regularity, and finally, whereas the principal part in the operator is of order $3/2$, the subprincipal part in the operator is of order 1 which gives only a $1/2$ difference compared to the usual 1 difference encountered for magnetic Schrödinger operators. As will be shown in our analysis, the presence of such subprincipal parts will produce non trivial oscillations which here have to be taken into account in the analysis.

The first common step for both theorems is to perform several reductions for the paradifferential equation. The first one is to use Alinhac's para-composition theory [4] (see also Burq-Planchon [14] where a similar idea was used) to reduce the matters to the study of a Schrödinger type operator with constant coefficients principal part. This is particular to space dimension 1 and reflects the fact that there is only one metric on $\mathbf{R}$. The second reduction, inspired by works by Smith [26], Bahouri-Chemin [7], Tataru [28, 29] and Blair [11], consists in smoothing out the coefficients of the operator.

Once this reduction has been achieved, we can construct the parametrix, for which the natural time is the semi-classical one: $s = t|\xi|^{-1/2}$. Here the differences between our