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ON THE INFINITE FERN OF GALOIS REPRESENTATIONS OF UNITARY TYPE

BY Gaëtan CHENEVIER

To Julia and Valeria

Abstract. – Let E be a CM number field, p an odd prime totally split in E, and let X be the p-adic analytic space parameterizing the isomorphism classes of 3-dimensional semisimple p-adic representations of $\mathrm{Gal}(\overline{E}/E)$ satisfying a selfduality condition "of type $\mathrm{U}(3)$". We study an analogue of the infinite fern of Gouvêa-Mazur in this context and show that each irreducible component of the Zariski-closure of the modular points in X has dimension at least $3[E : \mathbb{Q}]$. As important steps, and in any rank, we prove that any first order deformation of a generic enough crystalline representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is a linear combination of trianguline deformations, and that unitary eigenvarieties are étale over weight space at the non-critical classical points. As another application, we give a surjectivity criterion for the localization at p of the *adjoint′* Selmer group* of a p-adic Galois representation attached to a cuspidal cohomological automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$ of type $\mathrm{U}(n)$ (for any n).

Résumé. – Soient E un corps de nombres CM, p un nombre premier impair totalement décomposé dans E, et soit X l'espace analytique p-adique paramétrant les classes d'isomorphie de représentations p-adiques semisimples de dimension 3 de $\mathrm{Gal}(\overline{E}/E)$ satisfaisant une condition d'autodualité « de type $\mathrm{U}(3)$ ». Nous étudions un analogue de la fougère infinie de Gouvêa-Mazur dans ce contexte et démontrons que l'adhérence Zariski des points modulaires de X a toutes ses composantes irréductibles de dimension au moins $3[E : \mathbb{Q}]$. Au passage, nous prouvons en toute dimension que toute déformation à l'ordre 1 d'une représentation cristalline suffisamment générique de $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ est une combinaison linéaire de déformations triangulines, et que les variétés de Hecke unitaires sont étales sur l'espace des poids aux points classiques non critiques. Enfin, nous obtenons un critère de surjectivité de l'application de localisation en p du groupe de Selmer *adjoint′* d'une représentation galoisienne p-adique attachée à une représentation automorphe cuspidale cohomologique de $\mathrm{GL}_n(\mathbb{A}_E)$ qui est de type $\mathrm{U}(n)$ (pour tout n).

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* Pronounce "adjoint primed Selmer group."

Introduction

Let E be a number field, p a prime number, S a finite set of places of E containing the places above p and ∞, $\mathrm{G}_{E,S}$ the Galois group of a maximal algebraic extension of E unramified outside S, and let $n \geq 1$ be an integer. We are interested in the set of n-dimensional, semisimple, continuous representations

$$\rho : \mathrm{G}_{E,S} \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$$

taken up to isomorphism. This set turns out to be the $\overline{\mathbb{Q}}_p$-points of a rigid analytic space $\mathfrak{X}$ (or $\mathfrak{X}_n$) over $\mathbb{Q}_p$ in a natural way(1). An interesting subset

$$\mathfrak{X}^{\mathrm{g}} \subset \mathfrak{X}(\overline{\mathbb{Q}}_p)$$

is the subset of representations which are *geometric*, in the sense they occur as a subquotient of $H^i_{\mathrm{et}}(X_{\overline{E}}, \overline{\mathbb{Q}}_p)(m)$ for some proper smooth algebraic variety X over E and some integers $i \geq 0$ and $m \in \mathbb{Z}$. There are several basic open questions that we can ask about $\mathfrak{X}$ and its locus $\mathfrak{X}^{\mathrm{g}}$; here are some of them:

Questions. – Does $\mathfrak{X}^{\mathrm{g}}$ possess some specific structure inside $\mathfrak{X}$? What can we say about its various closures in $\mathfrak{X}$ (for example, for the Zariski or the p-adic topologies)? What if we replace $\mathfrak{X}^{\mathrm{g}}$ by its subset $\mathfrak{X}^{\mathrm{c}}$ of ρ's which are crystalline at the places of E above p?

Regarding the first question, a trivial observation is that $\mathfrak{X}^{\mathrm{g}}$ is countable, as the set of isomorphism classes of algebraic varieties over E is countable, thus it certainly contains no analytic subset of $\mathfrak{X}$ of positive dimension. In the simplest case $E = \mathbb{Q}$ and $S = \{\infty, p\}$ then $\mathfrak{X}_1$ is the space of p-adic continuous characters of $\mathbb{Z}_p^*$ (a finite union of 1-dimensional open balls) and $\mathfrak{X}^{\mathrm{c}}$ is the subset of characters of the form $x \mapsto x^m$ for $m \in \mathbb{Z}$, which is Zariski-dense in $\mathfrak{X}_1$. For a general E and S, we leave as an exercise to the reader to check that class field theory and the theory of complex multiplication show that $\mathfrak{X}^{\mathrm{c}}$ is also Zariski-dense in $\mathfrak{X}_1$ assuming Leopoldt's conjecture for E at p.

As a second and much more interesting example, let us recall the discovery of Gouvêa and Mazur [33]. They assume that $d = 2$, $E = \mathbb{Q}$, and say $S = \{\infty, p\}$ to simplify. Let q be a power of an odd prime p and let

$$\bar{\rho} : \mathrm{G}_{\mathbb{Q},S} \longrightarrow \mathrm{GL}_2(\mathbb{F}_q)$$

be an absolutely irreducible odd Galois representation. Let $R(\bar{\rho})$ denote the universal odd $\mathrm{G}_{\mathbb{Q},S}$-deformation ring of $\bar{\rho}$ in the sense of Mazur and let $\mathfrak{X}(\bar{\rho})$ be its analytic generic fiber: it is the connected component of $\mathfrak{X}_2$ parameterizing the ρ with residual representation $\bar{\rho}$. In general $\mathfrak{X}(\bar{\rho})$ is a rather complicated space, and Mazur first studied it in the *unobstructed case*(2) $H^2(\mathrm{G}_{\mathbb{Q},S}, \mathrm{ad}(\bar{\rho})) = 0$, for which class field theory shows that $R(\bar{\rho}) \simeq \mathbb{Z}_q[[x, y, z]]$, hence $\mathfrak{X}(\bar{\rho})$ is the open unit ball of dimension 3 over $\mathbb{Q}_q$. In this case, Gouvêa and Mazur showed that $\mathfrak{X}^{\mathrm{c}}$ *is Zariski-dense in* $\mathfrak{X}(\bar{\rho})$. They actually show that the subset

$$\mathfrak{X}^{\mathrm{mod}} \subset \mathfrak{X}(\bar{\rho})$$

(1) We will not use this space in the sequel, but for its definition see [13, Thm D].

(2) The philosophy of special values of L-functions suggests that this unobstructed case is the generic situation. It is now known for instance that for $\bar{\rho} = \bar{\rho}_\Delta$ attached to Ramanujan's $\Delta = \sum_{n>0} \tau(n) q^n$, $p > 13$ and $p \neq 691$, the deformation problem of $\bar{\rho}$ is unobstructed (Mazur, Weston [39]).

of p-adic Galois representations $\rho_f(m)$ attached to an eigenform $f \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ for some weight k, and some $m \in \mathbb{Z}$, is Zariski-dense in $\mathfrak{X}(\bar{\rho})$. This subset is non empty as $\bar{\rho}$ is modular (Khare-Wintenberger).

Their proof relies heavily on the theory of p-adic families of modular eigenforms due to Coleman, extending pioneering works of Hida, that we briefly recall. Let $f = q + a_2q^2 + a_3q^3 + \cdots \in \overline{\mathbb{Q}}_p[[q]]$ be a classical modular eigenform of level 1, of some weight k, and such that $\bar{\rho}_f = \bar{\rho}$; the representation ρ_f corresponds to some $x_f \in \mathfrak{X}(\bar{\rho})(\overline{\mathbb{Q}}_p)$. Attached to f we have two p-Weil numbers of weight $k-1$ which are the roots of the polynomial $X^2 - a_pX + p^{k-1}$. The main result of Coleman asserts that if φ is one of them, we can attach to (f, φ) an affinoid (equidimensional) subcurve

$$C_{(f,\varphi)} \subset \mathfrak{X}(\bar{\rho})$$

containing x_f as well as a Zariski-dense subset of modular points, and such that for each modular point $x_{f'} \in C_{(f,\varphi)}$ then f' has a p-Weil number with the same p-adic valuation as φ. This curve $C_{(f,\varphi)}$ is not quite canonical but its germ at x_f is. The infinite fern of Gouvêa-Mazur is by definition the union of all the $C_{(f,\varphi)}$ for all f and choice of φ. A simple but important observation made by Gouvêa and Mazur is that in some neighborhood of x_f in $\mathfrak{X}(\bar{\rho})$ the two curves $C_{(f,\varphi)}$ and $C_{(f,\varphi')}$ only intersect at the point x_f if φ and φ' have different valuations: this essentially follows from the previous properties and the fact that the "weight" varies analytically in Coleman's families. From this "fractal" picture it follows at once that the Zariski-closure of the modular points, or which is the same the Zariski-closure of the fern, has dimension at least 2 inside $\mathfrak{X}(\bar{\rho})$, and a simple argument of Tate-twists using Sen's theory gives then the result. The story does not quite end here as some years later, Coleman and Mazur defined a wonderful object, *the eigencurve*, which sheds new light on the infinite fern. They define a *refined modular point* as a pair

$$(x, \varphi) \in \mathfrak{X}(\bar{\rho}) \times \mathbb{G}_m$$

where $x = x_f$ is modular and φ is a p-Weil number of f. The interesting fact is that the Zariski-closure of the refined modular points in $\mathfrak{X}(\bar{\rho}) \times \mathbb{G}_m$ is an *equidimensional curve*, the so-called *$\bar{\rho}$-eigencurve*. Its image in $\mathfrak{X}(\bar{\rho})$ is the *complete infinite fern*, which simultaneously analytically continues each leaf of the infinite fern itself. This picture for $\mathfrak{X}(\bar{\rho})$ provides a rather satisfactory answer towards the first of the main questions above, even though very little is known about the geometry of the eigencurve at present. It is believed that $\mathscr{E}(\bar{\rho})$ has only finitely many irreducible components. An amazing consequence of this conjecture would be that for a well chosen modular point $x \in \mathfrak{X}(\bar{\rho})$ the analytic continuation of a well chosen leaf at x would be Zariski-dense in $\mathfrak{X}(\bar{\rho})$! However, as far as we know there is no non-trivial case in which this conjecture or its variants in other dimensions > 1 is known.

Let us mention that when S is general and $\bar{\rho}$ is possibly obstructed, the approach above of Gouvêa-Mazur still shows that each irreducible component of the Zariski-closure of the modular points in $\mathfrak{X}(\bar{\rho})$ has dimension at least 3. In a somehow opposite direction, it is conjectured that in all cases $\mathfrak{X}(\bar{\rho})$ is equidimensionnal of dimension 3, and that each of its irreducible components contains a smooth modular point. This conjecture, combined with the result of Gouvêa-Mazur, implies the Zariski-density of the modular points in $\mathfrak{X}(\bar{\rho})$ for each odd $\bar{\rho}$ (say absolutely irreducible). Relying on $R = T$ theorems of Taylor-Wiles, Diamond et al., Boeckle was able to show that conjecture under some rather mild assumption on $\bar{\rho}$, hence the Zariski-density in most cases: we refer to [6] for the precise statements.

Our main aim in this paper is to study a generalization of this picture to the higher dimensional case. Our most complete results will concern some pieces of $\mathfrak{X}_3$ satisfying some sort of self-duality condition. Let E be a CM field[(3)], and let

$$\bar{\rho} : \mathrm{G}_{E,S} \longrightarrow \mathrm{GL}_3(\mathbb{F}_q)$$

be an absolutely irreducible Galois representation such that $\bar{\rho}^* \simeq \bar{\rho}^c$, where c is a generator of $\mathrm{Gal}(E/F)$ and F the maximal totally real subfield of E. Let $\mathfrak{X}(\bar{\rho}) \subset \mathfrak{X}_3$ denote the closed subspace of $x \in \mathfrak{X}_3$ such that $\rho_x^* \simeq \rho_x^c$ and $\overline{\rho_x} \simeq \bar{\rho}$. This $\mathfrak{X}(\bar{\rho})$ has conjectural equidimension $6[F:\mathbb{Q}]$, and under an unobstructedness assumption similar to Mazur's one it is actually an open unit ball over $\mathbb{Q}_q$ in that number of variables. There is a natural notion of modular points in $\mathfrak{X}(\bar{\rho})$: they are the x such that ρ_x is a p-adic Galois representation attached to a cuspidal automorphic representation Π of $\mathrm{GL}_3(\mathbb{A}_E)$ such that Π_∞ is cohomological, $\Pi^\vee \simeq \Pi^c$, and such that for v finite dividing p or outside S, Π_v is unramified. Those Galois representations have been constructed by Rogawski; they are cut out from the étale cohomology of (some abelian varieties over) the Picard modular surfaces and they are related to automorphic forms on unitary groups in 3 variables associated to E/F.

THEOREM A. – *Assume that p is totally split in E. Then each irreducible component of the Zariski-closure of the modular points in $\mathfrak{X}(\bar{\rho})$ has dimension at least* $6[F:\mathbb{Q}]$.

In particular, in the unobstructed case the set of modular points of $\mathfrak{X}(\bar{\rho})$ is Zariski-dense if it is non-empty.

In the appendix, we give several examples of elliptic curves A over $\mathbb{Q}$ such that the deformation problem *of type* $U(3, E/\mathbb{Q})$ of $\bar{\rho} := \mathrm{Sym}^2 A[p]_{|\mathrm{G}_E}(-1)$ is unobstructed for $p = 5$ and $E = \mathbb{Q}(i)$. As in the work of [6] in the $\mathrm{GL}(2,\mathbb{Q})$ case, we expect that combining Theorem A with suitable $R = T$ theorems (as in [16] and subsequent work), one should be able to remove the unobstructedness assumption under suitable assumptions on $\bar{\rho}$. However, we postpone this to a subsequent study.

As in the work of Gouvêa-Mazur, a very important "constructive" ingredient of our proof is the theory of families of modular forms (for $U(3, E/F)$ here), or better, the related eigenvarieties. They can be quickly defined as follows, from the notion of *refined modular points*. Assume $F = \mathbb{Q}$ for simplicity and fix some prime v of E above p, so $E_v = \mathbb{Q}_p$ by assumption. For each modular point $x \in \mathfrak{X}(\bar{\rho})$, it is known that $\rho_{x,p} := \rho_{x|\mathrm{G}_{E_v}}$ is crystalline with distinct Hodge-Tate weights, say $k_1(x) < k_2(x) < k_3(x)$. Define a refined modular point as a pair $(x, (\widetilde{\varphi}_i(x))) \in \mathfrak{X}(\bar{\rho}) \times \mathbb{G}_m^3$ such that x is a modular point and such that

$$(p^{k_1(x)}\widetilde{\varphi}_1(x),\ p^{k_2(x)}\widetilde{\varphi}_2(x), p^{k_3(x)}\widetilde{\varphi}_3(x))$$

is an ordering of the eigenvalues of the crystalline Frobenius of $D_{\mathrm{crys}}(\rho_{x,p})$; there are up to 6 ways to refine a given modular point. We define the $\bar{\rho}$-eigenvariety

$$\mathcal{E}(\bar{\rho}) \subset \mathfrak{X}(\bar{\rho}) \times \mathbb{G}_m^3$$

(3) Throughout this paper, a CM field is assumed to be imaginary.