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RICCI FLOW COUPLED WITH HARMONIC MAP FLOW

BY Reto MÜLLER

ABSTRACT. – We investigate a coupled system of the Ricci flow on a closed manifold M with the harmonic map flow of a map ϕ from M to some closed target manifold N,

$$\frac{\partial}{\partial t} g = -2\mathrm{Rc} + 2\alpha \nabla\phi \otimes \nabla\phi, \qquad \frac{\partial}{\partial t}\phi = \tau_g \phi,$$

where α is a (possibly time-dependent) positive coupling constant. Surprisingly, the coupled system may be less singular than the Ricci flow or the harmonic map flow alone. In particular, we can always rule out energy concentration of ϕ a-priori by choosing α large enough. Moreover, it suffices to bound the curvature of $(M, g(t))$ to also obtain control of ϕ and all its derivatives if $\alpha \geq \underline{\alpha} > 0$. Besides these new phenomena, the flow shares many good properties with the Ricci flow. In particular, we can derive the monotonicity of an *energy*, an *entropy* and a *reduced volume* functional. We then apply these monotonicity results to rule out non-trivial breathers and geometric collapsing at finite times.

RÉSUMÉ. – Nous étudions un système d'équations consistant en un couplage entre le flot de Ricci et le flot harmonique d'une fonction ϕ allant de M dans une variété cible N,

$$\frac{\partial}{\partial t} g = -2\mathrm{Rc} + 2\alpha \nabla\phi \otimes \nabla\phi, \qquad \frac{\partial}{\partial t}\phi = \tau_g \phi,$$

où α est une constante de couplage strictement positive (et pouvant dépendre du temps). De manière surprenante, ce système couplé peut être moins singulier que le flot de Ricci ou le flot harmonique si ceux-ci sont considérés de manière isolée. En particulier, on peut toujours montrer que la fonction ϕ ne se concentre pas le long de ce système à condition de prendre α assez grand. De plus, il est suffisant de borner la courbure de $(M, g(t))$ le long du flot pour obtenir le contrôle de ϕ et de toutes ses dérivées si $\alpha \geq \underline{\alpha} > 0$. À part ces phénomènes nouveaux, ce flot possède certaines propriétés analogues à celles du flot de Ricci. En particulier, il est possible de montrer la monotonie d'une *énergie*, d'une *entropie* et d'une fonctionnelle *volume réduit*. On utilise la monotonie de ces quantités pour montrer l'absence de solutions en « accordéon » et l'absence d'effondrement en temps fini le long du flot.

1. Introduction and main results

Let (M^m, g) and (N^n, γ) be smooth Riemannian manifolds without boundary. According to Nash's embedding theorem [30] we can assume that N is isometrically embedded into Euclidean space $(N^n, \gamma) \hookrightarrow \mathbb{R}^d$ for a sufficiently large d. If $e_N : N \to \mathbb{R}^d$ denotes this embedding, we identify maps $\phi : M \to N$ with $e_N \circ \phi : M \to \mathbb{R}^d$, such maps may thus be written as $\phi = (\phi^\lambda)_{1 \leq \lambda \leq d}$. Harmonic maps $\phi : M \to N$ are critical points of the energy functional

$$E(\phi) = \int_M |\nabla\phi|^2 dV. \tag{1.1}$$

Here, $|\nabla\phi|^2 := 2e(\phi) = g^{ij}\nabla_i\phi^\lambda\nabla_j\phi^\lambda$ denotes the local energy density, where we use the convention that repeated Latin indices are summed over from 1 to m and repeated Greek indices are summed over from 1 to d. We often drop the summation indices for ϕ when clear from the context. Harmonic maps generalize the concept of harmonic functions and in particular include closed geodesics and minimal surfaces.

To study the existence of a harmonic map ϕ homotopic to a given map $\phi_0 : M \to N$, Eells and Sampson [13] proposed to study the L^2-gradient flow of the energy functional (1.1),

$$\tfrac{\partial}{\partial t}\phi = \tau_g\phi, \qquad \phi(0) = \phi_0, \tag{1.2}$$

where $\tau_g\phi$ denotes the intrinsic Laplacian of ϕ, often called the tension field of ϕ. They proved that if N has non-positive sectional curvature there always exists a unique, global, smooth solution of (1.2) which converges smoothly to a harmonic map $\phi_\infty : M \to N$ homotopic to ϕ_0 as $t \to \infty$ suitably. On the other hand, without an assumption on the curvature of N, the solution might blow up in finite or infinite time. Comprehensive surveys about harmonic maps and the harmonic map flow are given in Eells-Lemaire [11, 12], Jost [18] and Struwe [40]. The harmonic map flow was the first appearance of a nonlinear heat flow in Riemannian geometry. Today, geometric heat flows have become an intensely studied topic in geometric analysis.

Another fundamental problem in differential geometry is to find canonical metrics on Riemannian manifolds, for example metrics with constant curvature in some sense. Using the idea of evolving an object to such an ideal state by a nonlinear heat flow, Richard Hamilton [15] introduced the Ricci flow in 1982. His idea was to smooth out irregularities of the curvature by evolving a given Riemannian metric g on a manifold M with respect to the nonlinear weakly parabolic equation

$$\tfrac{\partial}{\partial t}g = -2\mathrm{Rc}, \qquad g(0) = g_0, \tag{1.3}$$

where Rc denotes the Ricci curvature of (M, g). Strictly speaking, the Ricci flow is not the gradient flow of a functional $\mathcal{F}(g) = \int_M F(\partial^2 g, \partial g, g)dV$, but in 2002, Perelman [31] showed that it is gradient-like nevertheless. He presented a new functional which may be regarded as an improved version of the Einstein-Hilbert functional $E(g) = \int_M R\, dV$, namely

$$\mathcal{F}(g, f) := \int_M \Big(R + |\nabla f|^2\Big)e^{-f} dV. \tag{1.4}$$

The Ricci flow can be interpreted as the gradient flow of $\mathcal{F}$ modulo a pull-back by a family of diffeomorphisms. Hamilton's Ricci flow has a successful history. Most importantly, Perelman's work [31, 33] led to a completion of Hamilton's program [16] and a complete proof of Thurston's geometrization conjecture [42] and (using a finite extinction result from Perelman [32] or Colding and Minicozzi [8, 9]) of the Poincaré conjecture [34]. Introductory surveys on the Ricci flow and Perelman's functionals can be found in the books by Chow and Knopf [6], Chow, Lu and Ni [7], Müller [27] and Topping [43]. More advanced explanations of Perelman's proof of the two conjectures are given in Cao and Zhu [3] Chow et al. [4, 5], Kleiner and Lott [19] and Morgan and Tian [25, 26]. A good survey on Perelman's work is also given in Tao [41].

The goal of this article is to study a coupled system of the two flows (1.2) and (1.3). Again, we let (M^m, g) and (N^n, γ) be smooth manifolds without boundary and with $(N^n, \gamma) \hookrightarrow \mathbb{R}^d$. Throughout this article, we will assume in addition that M and N are compact, hence closed. However, many of our results hold for more general manifolds.

Let $g(t)$ be a family of Riemannian metrics on M and $\phi(t)$ a family of smooth maps from M to N. We call $(g(t), \phi(t))_{t\in[0,T)}$ a solution to the coupled system of Ricci flow and harmonic map heat flow with coupling constant $\alpha(t)$, the $(RH)_\alpha$ flow for short, if it satisfies

$$(RH)_\alpha \qquad \begin{cases} \frac{\partial}{\partial t} g = -2\mathrm{Rc} + 2\alpha \nabla\phi \otimes \nabla\phi, \\ \frac{\partial}{\partial t}\phi = \tau_g \phi. \end{cases}$$

Here, $\tau_g\phi$ denotes the tension field of the map ϕ with respect to the evolving metric g, and $\alpha(t) \geq 0$ denotes a (time-dependent) coupling constant. Finally, $\nabla\phi \otimes \nabla\phi$ has the components $(\nabla\phi \otimes \nabla\phi)_{ij} = \nabla_i\phi^\lambda \nabla_j \phi^\lambda$. In particular, $|\nabla\phi|^2$ as defined above is the trace of $\nabla\phi \otimes \nabla\phi$ with respect to g.

The special case where $N \subseteq \mathbb{R}$ and $\alpha \equiv 2$ was studied by List [22], his motivation coming from general relativity and the study of Einstein vacuum equations. Moreover, List's flow also arises as the Ricci flow of a warped product, see [28, Lemma A.3]. After completion of this work, we learned that another special case of $(RH)_\alpha$ with $N \subseteq SL(k\mathbb{R})/SO(k)$ arises in the study of the long-time behavior of certain Type III Ricci flows, see Lott [23] and a recent paper of Williams [44] for details and explicit examples.

The paper is organized as follows. In order to get a feeling for the flow, we first study explicit examples of solutions of $(RH)_\alpha$ as well as soliton solutions which are generalized fixed points modulo diffeomorphisms and scaling. The stationary solutions of $(RH)_\alpha$ satisfy $\mathrm{Rc} = \alpha\nabla\phi\otimes\nabla\phi$, where ϕ is a harmonic map. To prevent $(M, g(t))$ from shrinking to a point or blowing up, it is convenient to introduce a volume-preserving version of the flow.

In Section 3, we prove that for constant coupling functions $\alpha(t) \equiv \alpha > 0$ the $(RH)_\alpha$ flow can be interpreted as a gradient flow for an energy functional $\mathcal{F}_\alpha(g, \phi, f)$ modified by a family of diffeomorphisms generated by ∇f. If $(g(t), \phi(t))$ solves $(RH)_\alpha$ and e^{-f} is a solution to the adjoint heat equation under the flow, then $\mathcal{F}_\alpha$ is non-decreasing and constant if and only if $(g(t), \phi(t))$ is a gradient steady soliton. In the more general case where $\alpha(t)$ is a positive function, the monotonicity result still holds whenever $\alpha(t)$ is non-increasing. This section is based on techniques of Perelman [31, Section 1] for the Ricci flow.

In the fourth section, we prove short-time existence for the flow using again a method from Ricci flow theory known as DeTurck's trick (cf. [10]), i.e., we transform the weakly parabolic system $(RH)_\alpha$ into a strictly parabolic one by pushing it forward with a family of diffeomorphisms. Moreover, we compute the evolution equations for the Ricci and scalar curvature, the gradient of ϕ and combinations thereof. In particular, the evolution equations for the symmetric tensor $S_{ij} := R_{ij} - \alpha\nabla_i\phi\nabla_j\phi$ and its trace $S = R - \alpha|\nabla\phi|^2$ will be very useful.

In Section 5, we study first consequences of the evolution equations for the existence or non-existence of certain types of singularities. Using the maximum principle, we show that $\min_{x\in M} S(x,t)$ is non-decreasing along the flow. This has the rather surprising consequence that if $|\nabla\phi|^2(x_k,t_k) \to \infty$ for $t_k \nearrow T$, then $R(x_k,t_k)$ blows up as well, i.e., $g(t_k)$ must become singular as $t_k \nearrow T$. Conversely, if $|\mathrm{Rm}|$ stays bounded along the flow, $|\nabla\phi|^2$ must stay bounded, too. This leads to the conjecture that a uniform Riemann-bound is enough to conclude long-time existence. This conjecture is proved in Section 6. To this end, we first compute estimates for the Riemannian curvature tensor, its derivatives and the higher derivatives of ϕ and then follow Bando's [1] and Shi's [38] results for the Ricci flow to derive interior-in-time gradient estimates.

In Section 7, we introduce an entropy functional $\mathcal{W}_\alpha(g,\phi,f,\tau)$ which corresponds to Perelman's shrinker entropy for the Ricci flow [31, Section 3]. Here $\tau = T - t$ denotes a backwards time. For $\alpha(t) \equiv \alpha > 0$, the entropy functional is non-decreasing and constant exactly on shrinking solitons. Again, the entropy is monotone if we allow non-increasing positive coupling functions $\alpha(t)$ instead of constant ones. Using $\mathcal{F}_\alpha$ and $\mathcal{W}_\alpha$ we can exclude nontrivial breathers, i.e., we show that a breather has to be a gradient soliton. In the case of a steady or expanding breather the result is even stronger, namely we can show that $\phi(t)$ has to be harmonic in these cases for all t.

Finally in the last section, we state the monotonicity of a backwards reduced volume quantity for the $(RH)_\alpha$ flow with positive non-increasing $\alpha(t)$. This follows from our more general result from [29]. We apply this monotonicity to deduce a local non-collapsing theorem.

In the appendix, we collect the commutator identities on bundles like $T^*M\otimes\phi^*TN$, which we need for the evolution equations in Sections 4 and 6.

This article originates from the author's PhD thesis [28] from 2009, where some of the proofs and computations are carried out in more details. The author likes to thank Klaus Ecker, Robert Haslhofer, Gerhard Huisken, Tom Ilmanen, Peter Topping and in particular Michael Struwe for stimulating discussions and valuable remarks and suggestions while studying this new flow. Moreover, he thanks the Swiss National Science Foundation that partially supported his research and Zindine Djadli who translated the abstract into flawless French.

2. Examples and special solutions

In this section, we only consider time-independent coupling constants $\alpha(t) \equiv \alpha$. First, we study two very simple homogeneous examples for the $(RH)_\alpha$ flow system to illustrate the different behavior of the flow for different coupling constants α. In particular, the existence or non-existence of singularities will depend on the choice of α. We study the volume-preserving