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LOCALLY ANALYTIC VECTORS OF UNITARY PRINCIPAL SERIES OF $\mathrm{GL}_2(\mathbb{Q}_p)$

BY RUOCHUAN LIU, BINGYONG XIE AND YUANCAO ZHANG

ABSTRACT. – The p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ attaches to any 2-dimensional irreducible p-adic representation V of $G_{\mathbb{Q}_p}$ an admissible unitary representation $\Pi(V)$ of $\mathrm{GL}_2(\mathbb{Q}_p)$. The unitary principal series of $\mathrm{GL}_2(\mathbb{Q}_p)$ are those $\Pi(V)$ corresponding to trianguline representations. In this article, for $p > 2$, using the machinery of Colmez, we determine the space of locally analytic vectors $\Pi(V)_{\mathrm{an}}$ for all non-exceptional unitary principal series $\Pi(V)$ of $\mathrm{GL}_2(\mathbb{Q}_p)$ by proving a conjecture of Emerton.

RÉSUMÉ. – La correspondance de Langlands locale p-adique pour $\mathrm{GL}_2(\mathbb{Q}_p)$ associe à toute représentation irréductible p-adique V de dimension 2 de $G_{\mathbb{Q}_p}$ une représentation admissible unitaire $\Pi(V)$ de $\mathrm{GL}_2(\mathbb{Q}_p)$. Les séries principales unitaires de $\mathrm{GL}_2(\mathbb{Q}_p)$ sont les $\Pi(V)$ correspondant aux représentations triangulines. Dans le présent article, en utilisant la machinerie de Colmez, on détermine l'espace des vecteurs localement analytiques $\Pi(V)_{\mathrm{an}}$ pour toute série principale unitaire non-exceptionnelle $\Pi(V)$ de $\mathrm{GL}_2(\mathbb{Q}_p)$, et on démontre ainsi une conjecture d'Emerton.

1. Introduction

Let F be a finite extension of $\mathbb{Q}_p$. The aim of the p-adic local Langlands programme initiated by Breuil is to look for a "natural" correspondence between certain n-dimensional p-adic representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$ and certain Banach space representations of $\mathrm{GL}_n(F)$. Thanks to much recent work, especially that of Colmez and Paškūnas, we have gained a fairly clear picture in the case $F = \mathbb{Q}_p$ and $n = 2$ which is the so-called p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ establishing a functorial bijection between 2-dimensional irreducible p-adic representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and non-ordinary irreducible admissible unitary representations of $\mathrm{GL}_2(\mathbb{Q}_p)$.

Although the present version of p-adic local Langlands correspondence is formulated at the level of Banach space representations, it is very useful, as in Breuil's initial work [4], to have information of the subspace of locally analytic vectors. Fix a finite extension L of $\mathbb{Q}_p$ as the coefficient field, and we denote by $\Pi(V)$ the corresponding unitary representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ for any 2-dimensional irreducible L-linear representation V of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$.

The *unitary principal series of* $\mathrm{GL}_2(\mathbb{Q}_p)$, which are the simplest ones among these $\Pi(V)$, are those corresponding to trianguline representations. In [13], Emerton made a conjectural description of the subspace of locally analytic vectors $\Pi(V)_{\mathrm{an}}$ for all unitary principal series $\Pi(V)$. We recall his conjecture below.

Let $\mathscr{S}_{\mathrm{irr}}$ be the parameterizing space of 2-dimensional irreducible trianguline representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ introduced by Colmez in [7]. For any $s \in \mathscr{S}_{\mathrm{irr}}$, let $V(s)$ be the corresponding trianguline representation. We may write $s = (\delta_1, \delta_2, \mathscr{L})$ so that the étale (φ, Γ)-module $\mathrm{D}_{\mathrm{rig}}(V(s))$ is isomorphic to the extension of $\mathscr{R}(\delta_2)$ by $\mathscr{R}(\delta_1)$ defined by $\mathscr{L}$. For any such s, if $\delta_1\delta_2^{-1} = x^k|x|$ for some $k \in \mathbb{Z}_+$, then we set $\Sigma(s)$ to be the locally analytic $\mathrm{GL}_2(\mathbb{Q}_p)$-representation $\Sigma(k+1, \mathscr{L}) \otimes ((\delta_2|x|^{\frac{2-k}{2}}) \circ \det)$ where $\{\Sigma(k+1, \mathscr{L})\}$ is the family of locally analytic $\mathrm{GL}_2(\mathbb{Q}_p)$-representations introduced by Breuil in [4]. Otherwise, we define $\Sigma(s)$ to be the locally analytic principal series $\left(\mathrm{Ind}_{\mathrm{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \delta_2 \otimes \delta_1(x|x|)^{-1}\right)_{\mathrm{an}}$. The conjecture of Emerton is:

Conjecture 1.1 ([13, Conjecture 6.7.3, 6.7.7]). – *For any $s \in \mathscr{S}_{\mathrm{irr}}$, $\Pi(V(s))_{\mathrm{an}}$ sits in an exact sequence*

$$0 \longrightarrow \Sigma(s) \longrightarrow \Pi(V(s))_{\mathrm{an}} \longrightarrow \left(\mathrm{Ind}_{\mathrm{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \delta_1 \otimes \delta_2(x|x|)^{-1}\right)_{\mathrm{an}} \longrightarrow 0. \tag{1.1}$$

In the special cases when $V(s)$ are twists of crystabelian representations and *non-exceptional*, there is a more precise conjectural description of $\Pi(V(s))_{\mathrm{an}}$ due to Breuil. In [16], the first author proved Breuil's conjecture. The main result of this paper is:

Theorem 1.2 (Theorem 6.13). – *For $p > 2$, Conjecture 1.1 is true if $V(s)$ is non-exceptional.*

In fact, one can easily deduce Breuil's conjecture from Emerton's conjecture. Thus for $p > 2$, the above theorem covers the aforementioned result of the first author.

We now explain the strategy of the proof of Theorem 1.2. The whole proof builds on Colmez's machinery of p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ developed in [12]. The key ingredient is Colmez's identification of the locally analytic vectors:

$$(\Pi(\check{V})_{\mathrm{an}})^* = D_{\mathrm{rig}}^{\natural} \boxtimes \mathrm{P}^1 \tag{1.2}$$

where $D = \mathrm{D}(V)$ is Fontaine's étale (φ, Γ)-module associated to V. To apply this formula, for any continuous characters $\delta, \eta : \mathbb{Q}_p^\times \to L^\times$, we construct the objects $\mathscr{R}(\eta) \boxtimes_\delta \mathrm{P}^1$ and $\mathscr{R}^+(\eta) \boxtimes_\delta \mathrm{P}^1$ which are equipped with continuous $\mathrm{GL}_2(\mathbb{Q}_p)$-actions, and the latter is topologically isomorphic to $(\left(\mathrm{Ind}_{\mathrm{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \delta^{-1}\eta \otimes \eta^{-1}\right)_{\mathrm{an}})^*$. In doing so, we are led to modify some of Colmez's constructions to twists of étale (φ, Γ)-modules and rank 1 (φ, Γ)-modules. On the other hand, Colmez [9] (for $p > 2$ and $s \in \mathscr{S}_*^{\mathrm{ng}} \coprod \mathscr{S}_*^{\mathrm{st}}$; this is the only place where we need $p > 2$) and Berger-Breuil [3] (for $s \in \mathscr{S}_{\mathrm{irr}}$ non-exceptional) establish an explicit isomorphism $\mathscr{A}_s : \Pi(V(s)) \cong \Pi(s)$ (for s exceptional, Paškūnas proves $\Pi(V(s)) \cong \Pi(s)$ by an indirect method [17]) where $\Pi(s)$ is the unitary principal series associated to $V(s)$. We deduce from the explicit description of $\mathscr{A}_s$ plus a duality argument the following exact sequence

$$0 \longrightarrow \mathscr{R}(\delta_1) \boxtimes \mathrm{P}^1 \longrightarrow \mathrm{D}_{\mathrm{rig}}(V(s)) \boxtimes \mathrm{P}^1 \longrightarrow \mathscr{R}(\delta_2) \boxtimes \mathrm{P}^1 \longrightarrow 0. \tag{1.3}$$

Then the natural inclusion $\left(\mathrm{Ind}_{\mathrm{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\delta_2\otimes\delta_1(x|x|)^{-1}\right)_{\mathrm{an}}\hookrightarrow\Pi(s)_{\mathrm{an}}$ induces the following commutative diagram

$$\begin{tikzcd}
(\Pi(s)_{\mathrm{an}})^* \arrow[r] \arrow[d] & D^{\natural}_{\mathrm{rig}}(\check{s})\boxtimes \mathrm{P}^1 \arrow[d] \\
\left(\left(\mathrm{Ind}_{\mathrm{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\delta_2\otimes\delta_1(x|x|)^{-1}\right)_{\mathrm{an}}\right)^* \arrow[r] & \mathscr{R}(\check{\delta}_1)\boxtimes \mathrm{P}^1
\end{tikzcd} \tag{1.4}$$

where $\check{s}=(\check{\delta}_2,\check{\delta}_1,\mathscr{L})$ corresponds to the Tate dual of $V(s)$. Using (1.4) together with the fact that $D^{\natural}_{\mathrm{rig}}(s)\boxtimes\mathrm{P}^1$ and $D^{\natural}_{\mathrm{rig}}(\check{s})\boxtimes\mathrm{P}^1$ are orthogonal complements of each other under the paring

$$\{\cdot,\cdot\}_{\mathrm{P}^1}:\mathrm{D}_{\mathrm{rig}}(V(s))\boxtimes\mathrm{P}^1\times\mathrm{D}_{\mathrm{rig}}(V(\check{s}))\boxtimes\mathrm{P}^1\to L,$$

and that (1.3) is dual to

$$0\longrightarrow\mathscr{R}(\check{\delta}_2)\boxtimes\mathrm{P}^1\longrightarrow\mathrm{D}_{\mathrm{rig}}(V(\check{s}))\boxtimes\mathrm{P}^1\longrightarrow\mathscr{R}(\check{\delta}_1)\boxtimes\mathrm{P}^1\longrightarrow 0, \tag{1.5}$$

we deduce that if $\delta_1\delta_2^{-1}\neq x^k|x|$ for any $k\in\mathbb{Z}_+$, then $D^{\natural}_{\mathrm{rig}}(\check{s})$ sits in the exact sequence

$$0\longrightarrow\mathscr{R}^+(\check{\delta}_2)\boxtimes\mathrm{P}^1\longrightarrow D^{\natural}_{\mathrm{rig}}(\check{s})\boxtimes\mathrm{P}^1\longrightarrow\mathscr{R}^+(\check{\delta}_1)\boxtimes\mathrm{P}^1\longrightarrow 0. \tag{1.6}$$

We therefore conclude (1.1) by taking the dual of (1.6). If $\delta_1\delta_2^{-1}=x^k|x|$ for some $k\in\mathbb{Z}_+$, we have that the image of $D^{\natural}_{\mathrm{rig}}(\check{s})$ in $\mathscr{R}(\check{\delta}_1)\boxtimes\mathrm{P}^1$ is a closed subspace of $\mathscr{R}^+(\check{\delta}_1)\boxtimes\mathrm{P}^1$ of codimension k and $\mathscr{R}(\check{\delta}_2)\boxtimes\mathrm{P}^1\cap D^{\natural}_{\mathrm{rig}}(\check{s})\boxtimes\mathrm{P}^1$ contains $\mathscr{R}^+(\check{\delta}_2)\boxtimes\mathrm{P}^1$ as a closed subspace of codimension k. We then conclude (1.1) using Schneider and Teitelbaum's results on the Jordan-Hölder series of locally analytic principal series of $\mathrm{GL}_2(\mathbb{Q}_p)$.

The organization of the paper is as follows. In §2, we recall some necessary background of the theory of (φ,Γ)-modules. In §3, we recall some of Colmez's constructions of the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ especially his identification of the locally analytic vectors, and we make the aforementioned modification. We review the isomorphism $\mathscr{A}_s:\Pi(V(s))\cong\Pi(s)$ in §4. In §5.1, we recall Schneider and Teitelbaum's results on Jordan-Hölder series of the locally analytic principal series of $\mathrm{GL}_2(\mathbb{Q}_p)$. We prove that $\mathscr{R}^+(\eta)\boxtimes_\delta\mathrm{P}^1$ is isomorphic to $(\left(\mathrm{Ind}_{\mathrm{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\delta^{-1}\eta\otimes\eta^{-1}\right)_{\mathrm{an}})^*$ in §5.2. Section 6 is devoted to the proof of Theorem 1.2. We first recall the definition of $\Sigma(s)$ and restate Emerton's conjecture in §6.1. Then we prove the exact sequence (1.3) in §6.2. We finish the proof of Theorem 1.2 in §6.3.

After the work presented in this paper was finished, we learned from Colmez that he had a proof of Conjecture 1.1 for all p and all trianguline representations of $G_{\mathbb{Q}_p}$. The strategy of his proof is different from ours. He constructs the map $\Pi(s)_{\mathrm{an}}\to\left(\mathrm{Ind}_{\mathrm{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\delta_1\otimes\delta_2(x|x|)^{-1}\right)_{\mathrm{an}}$ directly by computing the *module de Jacquet dual* of $\Pi(s)_{\mathrm{an}}$. We refer the reader to his paper [10] for more details.

Notation and conventions

Let p be a prime number, and let v_p denote the p-adic valuation on $\overline{\mathbb{Q}}_p$, normalized by $v_p(p)=1$; the corresponding norm is denoted by $|\cdot|$. Let $G_{\mathbb{Q}_p}$ denote $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ for simplicity. Let $\chi:G_{\mathbb{Q}_p}\to\mathbb{Z}_p^\times$ be the p-adic cyclotomic character. The kernel of χ is $H=\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p(\mu_{p^\infty}))$, and let $\Gamma=\mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$. For any $a\in\mathbb{Z}_p^\times$, let σ_a be the unique element in Γ such that $\chi(\sigma_a) = a$. For any positive integer h, let $\Gamma_h = \chi^{-1}(1+p^h\mathbb{Z}_p)$. If we regard χ as a character of $\mathbb{Q}_p^\times$ via the local Artin map, then it is equal to $\epsilon(x) = x|x|$. Throughout this paper, we fix a finite extension L of $\mathbb{Q}_p$. We denote by $\mathcal{O}_L$ the ring of integers of L, and by k_L the residue field. Let $\widehat{\mathscr{T}}(L)$ be the set of all continuous characters $\delta : G_{\mathbb{Q}_p} \to L^\times$. For any $\delta \in \widehat{\mathscr{T}}(L)$, the Hodge-Tate weight $w(\delta)$ of δ is defined by $w(\delta) = \frac{\log \delta(u)}{\log u}$ where u is any element of $\mathbb{Q}_p^\times \setminus \mu_{p^\infty}$. For any L-linear representation V of $G_{\mathbb{Q}_p}$, we denote by $\check{V}$ the Tate dual $V^*(\epsilon)$ of V. For any $\mathscr{L} \in L$, let $\log_{\mathscr{L}} : \mathbb{Q}_p^\times \to L$ be the homomorphism defined by $\log_{\mathscr{L}}(p)$=1 and $\log_{\mathscr{L}}(x) = -\sum_{n=1}^{+\infty} \frac{(1-x)^n}{n}$ when $|x-1| < 1$. We put $\log_\infty = v_p$. Hence $\log_{\mathscr{L}}$ is defined for all $\mathscr{L} \in \mathrm{P}^1(L)$.

Let B be the subgroup of upper triangular matrices of GL_2, let $\mathrm{P} = \left[\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right]$ be the mirabolic subgroup of GL_2, let T be the subgroup of diagonal matrices of GL_2, and let Z be the center of GL_2. Put $w = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Let $\mathrm{Rep}_{\mathrm{tors}}\mathrm{GL}_2(\mathbb{Q}_p)$ be the category of smooth $\mathcal{O}_L[\mathrm{GL}_2(\mathbb{Q}_p)]$-modules which are of finite lengths and admit central characters. Let $\mathrm{Rep}_{\mathcal{O}_L}\mathrm{GL}_2(\mathbb{Q}_p)$ be the category of $\mathcal{O}_L[\mathrm{GL}_2(\mathbb{Q}_p)]$-modules Π which are separated and complete for the p-adic topology, p-torsion free, and $\Pi/p^n\Pi \in \mathrm{Rep}_{\mathrm{tors}}\mathrm{GL}_2(\mathbb{Q}_p)$ for any $n \in \mathbb{N}$.

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2. Preliminaries on (φ, Γ)-modules

2.1. Dictionary of p-adic functional analysis

Let $\mathcal{O}_{\mathscr{E}}$ be the ring of Laurent series $f = \sum_{i\in\mathbb{Z}} a_i T^i$, where $a_i \in \mathcal{O}_L$, such that $v_p(a_i) \to \infty$ as $i \to -\infty$. Let $\mathscr{E} = \mathcal{O}_{\mathscr{E}}[1/p]$ be the fraction field of $\mathcal{O}_{\mathscr{E}}$.

For any $r \in \mathbb{R}_+ \cup \{+\infty\}$, let $\mathscr{E}^{]0,r]}$ be the ring of Laurent series $f = \sum_{i\in\mathbb{Z}} a_i T^i$, with $a_i \in L$, such that f is convergent on the annulus $0 < v_p(T) \leq r$. For any $0 < s \leq r \leq +\infty$, we define the valuation $v^{\{s\}}$ on $\mathscr{E}^{]0,r]}$ by

$$v^{\{s\}}(f) = \inf_{i\in\mathbb{Z}}\{v_p(a_i) + is\} \text{ if } s \neq \infty;\ v^{\{\infty\}}(f) = v_p(f(0)).$$

We provide $\mathscr{E}^{]0,r]}$ with the Fréchet topology defined by the family of valuations $\{v^{\{s\}} | 0 < s \leq r\}$; then $\mathscr{E}^{]0,r]}$ is complete. We equip the *Robba ring* $\mathscr{R} = \bigcup_{r>0} \mathscr{E}^{]0,r]}$