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ROBUST TRANSITIVITY IN HAMILTONIAN DYNAMICS

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ABSTRACT. – A goal of this work is to study the dynamics in the complement of KAM tori with focus on non-local robust transitivity. We introduce C^r open sets ($r = 1, 2, \dots, \infty$) of symplectic diffeomorphisms and Hamiltonian systems, exhibiting *large* robustly transitive sets. We show that the C^∞ closure of such open sets contains a variety of systems, including so-called *a priori* unstable integrable systems. In addition, the existence of ergodic measures with large support is obtained for all those systems. A main ingredient of the proof is a combination of studying minimal dynamics of symplectic iterated function systems and a new tool in Hamiltonian dynamics which we call "symplectic blender".

RÉSUMÉ. – Un objectif de ce travail est d'étudier la dynamique sur le complémentaire des tores KAM en mettant l'accent sur la transitivité robuste non locale. Nous introduisons les ensembles ouverts de difféomorphismes symplectiques et de systèmes hamiltoniens, présentant de grands ensembles robustement transitifs. L'adhérence de ces ensembles ouverts (en topologie C^r, $r = 1, 2, \dots, \infty$) contient un grand nombre de systèmes, y compris les systèmes intégrables *a priori* instables. En outre, l'existence de mesures ergodiques avec un grand support est obtenue pour l'ensemble de ces systèmes. L'ingrédient principal des preuves est la combinaison de l'étude de systèmes itérés de fonctions de dynamique minimale et d'un nouvel outil de la dynamique hamiltonienne que nous appelons « mélangeurs symplectiques ».

1. Introduction and main results

The theory of Kolmogorov, Arnold and Moser (KAM) gives a precise description of the dynamics of a set of large measure of orbits for any small perturbation of a non-degenerate integrable Hamiltonian system. These orbits lie on the invariant KAM tori for which the dynamics are equivalent to irrational (Diophantine) rotations. In the case of autonomous systems in two degrees of freedom or time-periodic systems in one degree of freedom (i.e., 1.5 degree of freedom), the KAM Theorem proves the stability of *all* orbits, in the sense that the action variable does not vary much along the orbits. This, of course, is not the case if the degree of freedom is larger than two, where the KAM tori has codimension of at least two. A natural question arises: *Do generic perturbations of integrable systems in higher dimensions* *exhibit instabilities?* The first example of instability is due to Arnold [4], who constructed a family of small perturbations of a non-degenerate integrable Hamiltonian system that exhibits instability in the sense that there are orbits with large action variation. This kind of topological instability is sometimes called the *Arnold diffusion*. Indeed, it was conjectured [3, p. 176] that instability is a common phenomenon in the complement of integrable systems. Aside the several deep contributions towards this conjecture, especially in recent years (see e.g. [12], [17], [14], [21], [22], [28], [27], [37], and references therein), it is still one of the central problems in Hamiltonian dynamics.

Here, we would like to suggest a different approach related to the instability problem. We propose to focus on the existence and abundance of a dynamical phenomenon, more sophisticated than instability, which is *"large" robustly transitive sets*. Roughly speaking, a set is transitive if it contains a dense orbit inside, and it is robustly transitive if the same holds for all nearby systems (see Definitions 1.6, 1.8).

The present paper is devoted to studying the non-local robust transitivity (global or non-global) in symplectic and Hamiltonian dynamics with the goal of better understanding the dynamics in the complement of KAM tori, and with application to the instability problem.

In the non-conservative context, there are many important recent contributions about robust transitivity. Note that a diffeomorphism of a manifold M is transitive if it has a dense orbit in the whole manifold. Such a diffeomorphism is called C^r robustly transitive if it belongs to the C^r interior of the set of transitive diffeomorphisms. It has been known since the 1960's that any (transitive) hyperbolic diffeomorphism is C^1 robustly transitive. The first examples of non-hyperbolic C^1 robustly transitive sets are credited to M. Shub [36] and R. Mañé [25]. For a long time their examples remained unique. Then, C. Bonatti and L. Díaz [7] introduced a semi-local source for transitivity, called *blender*, which is C^1 robust. Using this tool they constructed new examples of robustly transitive sets and diffeomorphisms. For recent results involving blenders, see [8]. For the recent surveys on this topic and robust transitivity on compact manifolds, see [10, Chapters 7,8], [33], [32].

In this paper, we develop the methods of robust transitivity within the context of symplectic and Hamiltonian systems. We apply them for the nearly integrable symplectic and Hamiltonian systems with more than two degrees of freedom. Following this approach, we introduce open sets of such Hamiltonian or symplectic diffeomorphisms exhibiting *large* robustly transitive sets and containing integrable systems in their closure. Then, the instability (Arnold diffusion) is obtained as a consequence of the existence of large robustly transitive sets. We want to point out that the results obtained also include systems not necessarily close to integrable ones.

We also obtain good information about the structure and dynamics of the robustly transitive sets that yield to topological mixing and even ergodicity. These are the scope of theorems stated in Sections 1.2 - 1.6.

We would like to compare the usual notion of instability (i.e. Arnold diffusion as treated in [14, 12, 37]) with robust transitivity (or topological mixing) obtained in the thesis of our theorems. Observe that the usual notion of instability is a C^0 robust property since it depends only on a finite number of iterations. However, there are no topologically mixing or transitive systems which are C^0 robust (see also Section 6.2).

Let us emphasize that in this paper we deal with the C^r-topology for any $r = 1, \dots, \infty$. We also work with non-compact manifolds.

Section 1.1 introduces some definitions and notations; in Sections 1.2–1.6 the main theorems are stated. The two main ingredients used in the proofs are described informally in Sections 1.7 and 1.8. Finally, Section 1.9 provides a heuristic explanation of how these ingredients are combined and used.

1.1. Preliminaries and definitions

Some of the definitions below are standard in the literature so we only highlight the ones that are not common.

Let M be a boundaryless Riemannian manifold (not necessarily compact) and $f : M \longrightarrow M$ be a C^r diffeomorphism of a manifold M. From now on we assume that $r \in [1, \infty]$. We denote by $\mathrm{Diff}^r(M)$ the space of C^r diffeomorphisms of M endowed with the uniform C^r topology.

An f-invariant subset Λ is *partially hyperbolic* if its tangent bundle $T_\Lambda M$ splits as a Whitney sum of Df-invariant subbundles:

$$T_\Lambda M = E^u \oplus E^c \oplus E^s,$$

and there exist a Riemannian metric on M, a positive integer n_0 and constants $0 < \lambda < 1$ and $\mu > 1$ such that for every $p \in \Lambda$,

$$0 < \| D_p f^{n_0}|_{E^s} \| < \lambda < m(D_p f^{n_0}|_{E^c}) \leq \| D_p f^{n_0}|_{E^c} \| < \mu < m(D_p f^{n_0}|_{E^u}).$$

The co-norm $m(A)$ of a linear operator A between Banach spaces is defined by $m(A) := \inf\{\| A(v) \| : \| v \| = 1\}$. The subbundles E^u, E^c and E^s are referred to the unstable, center and stable bundles of f, respectively.

A partially hyperbolic set is called *hyperbolic* if its center bundle is trivial, i.e. $E^c = \{0\}$.

Definition 1.1 (domination). – Let f and g be two diffeomorphisms on manifolds M and N respectively. Suppose that $\Lambda \subset M$ is an invariant hyperbolic set for f. We say that g is dominated by $f|_\Lambda$ if $\Lambda \times N$ is a partially hyperbolic set for $f \times g$, with $E^c = TN$.

The *homoclinic class of a hyperbolic set* is the closure of the transversal intersections of its stable and unstable manifolds. In the case of a hyperbolic periodic point P of a diffeomorphism F, we denote its homoclinic class by $H(P, F)$. Moreover, for any G nearby F, we denote by P_G the *analytic continuation* of P and by $H(P_G, G)$ its homoclinic class.

Definition 1.2 (weak hyperbolic point). – Let p be a hyperbolic periodic point of g of period k; we say that p is δ-*weak hyperbolic* if

$$1 - \delta < m(D_p g^k|_{E^s_p}) < \| D_p g^k|_{E^s_p} \| < 1 < m(D_p g^k|_{E^u_p}) < \| D_p g^k|_{E^u_p} \| < \frac{1}{1-\delta}.$$

Let X be a metric space and $F : X \to X$ a continuous transformation. A set $Y \subset X$ (not necessarily compact) is *transitive* for F if for any U_1, U_2 open in X, such that $U_i \cap Y \neq \varnothing$, there is some n with $F^n(U_1) \cap U_2 \neq \varnothing$. If in addition, for any open sets $U_1, U_2 \subset Y$ (in the restricted topology), there is some n with $F^n(U_1) \cap U_2 \neq \varnothing$, then we say Y is *strictly transitive*. A stronger property is *topological mixing*, where $F^n(U_1) \cap U_2 \neq \varnothing$ holds for *any* sufficiently large n. Similarly we define *strictly topologically mixing*.

In the next definitions we denote by $\mathcal{D}^r$ a subspace of $\mathrm{Diff}^r(M)$ with the C^r topology.

Definition 1.3 (continuation). – A set $X \subset M$ of f has *continuation* in $\mathcal{D}^r$ if there exist an open neighborhood $\mathcal{U}$ of f in $\mathcal{D}^r$ and a continuous map $\Phi : \mathcal{U} \to \mathcal{P}(M)$ such that $\Phi(f) = X$, where $\mathcal{P}(M)$ is the space of all subsets of M with the Hausdorff topology. Then, $\Phi(g)$ is called the *continuation* of X for g.

Remark 1.4. – Note that it is not assumed that the continuation is neither homeomorphic to the initial set nor invariant. Compare with Definition 4.4.

Definition 1.5 (exceptional set). – Let Λ be a partially hyperbolic set. We say that X is an *exceptional subset* of Λ if $X \subset \Lambda$ and for any central leaf L of Λ, the closure of $X \cap L$ in L has zero Lebesgue measure in L.

Definition 1.6 (large set). – We say that a set X contained in Λ is large inside Λ if the Hausdorff distance of Λ and the interior of X in Λ is small.

Definition 1.7 (compact robustly transitive set). – A compact set $Y \subset M$ is $\mathcal{D}^r$ *robustly (strictly) transitive* for $f \in \mathcal{D}^r$, if for any $g \in \mathcal{D}^r$ sufficiently close to f, the continuation of Y does exist and it is (strictly) transitive for g. In the same way one may define robustly (strictly) topologically mixing.

Definition 1.8 (non-compact robustly transitive set). – If Y is not compact, then Y is called $\mathcal{D}^r$ *robustly (strictly) transitive* if it is the union of an increasing sequence of compact $\mathcal{D}^r$ robustly (strictly) transitive sets. In the same way one may define robustly (strictly) topologically mixing for non-compact sets.

A periodic point p of f of period n is called *quasi-elliptic* if D_pf^n has a non-real eigenvalue of norm one, and all eigenvalues of norm one are non-real. If in addition all eigenvalues have norm one then it is called *elliptic*.

A point x is *non-wandering* for a diffeomorphism f if for any neighborhood U of x there is $n \in \mathbb{N}$ such that $f^n(U) \cap U \neq \varnothing$. By $\Omega(f)$ we denote the set of all non-wandering point of f. A point x is called (positively) *recurrent* for f if $\liminf_{n\to+\infty} \mathrm{dist}(x, f^n(x)) = 0$. A diffeomorphism is said *recurrent* if Lebesgue almost all points are recurrent.