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ON THE PICARD NUMBER OF DIVISORS IN FANO MANIFOLDS

BY Cinzia CASAGRANDE

Abstract. – Let X be a complex Fano manifold of arbitrary dimension, and D a prime divisor in X. We consider the image $\mathcal{N}_1(D,X)$ of $\mathcal{N}_1(D)$ in $\mathcal{N}_1(X)$ under the natural push-forward of 1-cycles. We show that $\rho_X - \rho_D \leq \operatorname{codim} \mathcal{N}_1(D,X) \leq 8$. Moreover if $\operatorname{codim} \mathcal{N}_1(D,X) \geq 3$, then either $X \cong S \times T$ where S is a Del Pezzo surface, or $\operatorname{codim} \mathcal{N}_1(D,X) = 3$ and X has a fibration in Del Pezzo surfaces onto a Fano manifold T such that $\rho_X - \rho_T = 4$.

Résumé. – Soient X une variété de Fano lisse et complexe de dimension arbitraire, et D un diviseur premier dans X. Nous considérons l'image $\mathcal{N}_1(D,X)$ de $\mathcal{N}_1(D)$ dans $\mathcal{N}_1(X)$ par l'application naturelle de *push-forward* de 1-cycles. Nous démontrons que $\rho_X - \rho_D \leq \operatorname{codim} \mathcal{N}_1(D,X) \leq 8$. De plus, si $\operatorname{codim} \mathcal{N}_1(D,X) \geq 3$, alors soit $X \cong S \times T$ où S est une surface de Del Pezzo, soit $\operatorname{codim} \mathcal{N}_1(D,X) = 3$ et X a une fibration en surfaces de Del Pezzo sur une variété de Fano lisse T, telle que $\rho_X - \rho_T = 4$.

1. Introduction

Let X be a complex Fano manifold of arbitrary dimension n, and consider a prime divisor $D \subset X$. We denote by $\mathcal{N}_1(X)$ the $\mathbb{R}$-vector space of one-cycles in X, with real coefficients, modulo numerical equivalence; its dimension is the *Picard number* ρ_X of X, and similarly for D. The inclusion $i\colon D \hookrightarrow X$ induces a push-forward of one-cycles $i_*\colon \mathcal{N}_1(D) \to \mathcal{N}_1(X)$, that does not need to be injective nor surjective. We are interested in the image

$$\mathcal{N}_1(D,X) := i_*(\mathcal{N}_1(D)) \subseteq \mathcal{N}_1(X),$$

which is the linear subspace of $\mathcal{N}_1(X)$ spanned by numerical classes of curves contained in D. The codimension of $\mathcal{N}_1(D,X)$ in $\mathcal{N}_1(X)$ is equal to the dimension of the kernel of the restriction $H^2(X,\mathbb{R}) \to H^2(D,\mathbb{R})$.

If X is a surface, then it follows from the classification of Del Pezzo surfaces that $\operatorname{codim} \mathcal{N}_1(D,X) = \rho_X - 1 \leq 8$. Our main result is that the same holds in any dimension.

THEOREM 1.1. – *Let X be a Fano manifold of dimension n. For every prime divisor $D \subset X$, we have*

$$\rho_X - \rho_D \leq \operatorname{codim} \mathcal{N}_1(D, X) \leq 8.$$

Moreover, suppose that there exists a prime divisor D with $\operatorname{codim} \mathcal{N}_1(D, X) \geq 3$. *Then one of the following holds:*

(i) *$X \cong S \times T$, where S is a Del Pezzo surface with $\rho_S \geq \operatorname{codim} \mathcal{N}_1(D, X) + 1$, and D dominates T under the projection;*

(ii) $\operatorname{codim} \mathcal{N}_1(D, X) = 3$ *and there exists a flat surjective morphism $\varphi\colon X \to T$, with connected fibers, where T is an $(n-2)$-dimensional Fano manifold, and $\rho_X - \rho_T = 4$.*

When $n \geq 4$ and D is ample, one has $\mathcal{N}_1(D, X) = \mathcal{N}_1(X)$ and also $\dim \mathcal{N}_1(D, X) = \rho_D$ by Lefschetz Theorems on hyperplane sections, see [17, Example 3.1.25]. However in general $\dim \mathcal{N}_1(D, X)$ can be smaller than ρ_X: for instance, if $D \cong \mathbb{P}^{n-1}$ is the exceptional divisor of the blow-up X of any projective manifold at a point, we have $\rho_D = \dim \mathcal{N}_1(D, X) = 1 < \rho_X$.

In case (ii) of Theorem 1.1 the variety X does not need to be a product of lower dimensional varieties, see Example 3.4.

Theorem 1.1 generalizes an analogous result in [9] for toric Fano varieties, obtained in a completely different way, using combinatorial techniques.

We recall that the pseudo-index of a Fano manifold X is

$$\iota_X = \min\{-K_X \cdot C \mid C \text{ is a rational curve in } X\},$$

and is a multiple of the index of X; one expects that Fano manifolds with large pseudo-index are simpler. When $\iota_X > 1$ (*i.e.*, when X does not contain rational curves of anticanonical degree one), we show a stronger version of Theorem 1.1.

THEOREM 1.2. – *Let X be a Fano manifold with pseudo-index $\iota_X > 1$. For every prime divisor $D \subset X$, we have* $\operatorname{codim} \mathcal{N}_1(D, X) \leq 1$. *More precisely, one of the following holds:*

(i) *$\iota_X = 2$ and there exists a smooth morphism $\varphi\colon X \to Y$ with fibers isomorphic to $\mathbb{P}^1$, where Y is a Fano manifold with $\iota_Y > 1$;*

(ii) *for every prime divisor $D \subset X$, we have $\mathcal{N}_1(D, X) = \mathcal{N}_1(X)$, $\rho_X \leq \rho_D$, and the restriction $H^2(X, \mathbb{R}) \to H^2(D, \mathbb{R})$ is injective. Moreover for every pair of prime divisors D_1, D_2 in X, we have $D_1 \cap D_2 \neq \varnothing$.*

The author was led to this subject by the study of Fano manifolds with large Picard number (see [10] for an account of this problem). Let us mention two straightforward consequences of Theorem 1.1, which give bounds on ρ_X in some good situations. The first concerns the case $\dim X \leq 5$, while the second is about Fano manifolds having a morphism onto a curve.

COROLLARY 1.3. – *Let X be a Fano manifold, and suppose that there exists a prime divisor $D \subset X$ such that* $\operatorname{codim} \mathcal{N}_1(D, X) \geq 3$.

If $\dim X = 4$ *then either $\rho_X \leq 6$, or X is a product of Del Pezzo surfaces and $\rho_X \leq 18$.*

If $\dim X = 5$ *then either $\rho_X \leq 9$, or X is a product and $\rho_X \leq 19$.*

COROLLARY 1.4. – *Let X be a Fano manifold, $\varphi\colon X \to \mathbb{P}^1$ a surjective morphism with connected fibers, and $F \subset X$ a general fiber. Then $\rho_X \leq \rho_F + 8$.*

Moreover if $\rho_X \geq \rho_F + 4$, then $X \cong S \times T$ where S is a Del Pezzo surface, φ factors through the projection $X \to S$, and $F \cong \mathbb{P}^1 \times T$.

Finally, we notice that some of the properties given by Theorem 1.1 are inherited by varieties dominated by a Fano manifold. We give two applications, and refer the reader to Lemma 4.1 for a more general statement.

COROLLARY 1.5. – *Let X be a Fano manifold and $\varphi\colon X \to Y$ a surjective morphism. Suppose that there exists a prime divisor $D \subset X$ such that* $\dim \varphi(D) \leq 1$ *(this always holds if* $\dim Y = 2$*). Then $\rho_Y \leq 9$.*

Moreover if $\rho_Y \geq 5$ then $\dim Y \leq 3$ *and $X \cong S \times T$, where S is a Del Pezzo surface.*

COROLLARY 1.6. – *Let X be a Fano manifold and $\varphi\colon X \to Y$ a surjective morphism with* $\dim Y = 3$*. Then $\rho_Y \leq 10$.*

Moreover if $\rho_Y \geq 6$ then $X \cong S \times T$ where S is a Del Pezzo surface, T has a contraction onto $\mathbb{P}^1$, and φ factors through $X \to S \times \mathbb{P}^1$.

Outline of the paper

The idea that a special divisor should affect the geometry of X is classical. In [6] Fano manifolds containing a divisor $D \cong \mathbb{P}^{n-1}$ with normal bundle $\mathcal{N}_{D/X} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(-1)$ are classified. This classification has been extended in [20] to the case $\mathcal{N}_{D/X} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(-a)$ with $a > 0$; moreover [20, Proposition 5] shows that if X contains a divisor D with $\rho_D = 1$, then $\rho_X \leq 3$. More generally, divisors $D \subset X$ with $\dim \mathcal{N}_1(D, X) = 1$ or 2 play an important role in [10, 11].

In Section 2 we treat the main construction that will be used in the paper, based on the analysis of a Mori program for $-D$, where $D \subset X$ is a prime divisor; this is a development of a technique used in [11]. Let us give an idea of our approach, referring the reader to Section 2 for more details.

After [5, 13], we know that we can run a Mori program for any divisor in a Fano manifold X. In fact we need to consider *special Mori programs*, where all involved extremal rays have positive intersection with the anticanonical divisor (see Section 2.1).

Then, given a prime divisor $D \subset X$, we consider a special Mori program for $-D$, which roughly means that we contract or flip extremal rays having positive intersection with D, until we get a fiber type contraction such that (the transform of) D dominates the target.

If $c := \operatorname{codim} \mathcal{N}_1(D, X) > 0$, by studying how the codimension of $\mathcal{N}_1(D, X)$ varies under the birational maps and the related properties of the extremal rays, we obtain $c - 1$ *pairwise disjoint* prime divisors $E_1, \dots, E_{c-1} \subset X$, all intersecting D, such that each E_i is a smooth $\mathbb{P}^1$-bundle with $E_i \cdot f_i = -1$, where $f_i \subset E_i$ is a fiber (see Proposition 2.5 and Lemma 2.7). We call $E_1, \dots, E_{c-1}$ the $\mathbb{P}^1$-bundles determined by the special Mori program for $-D$ that we are considering; they play an essential role throughout the paper.

We conclude Section 2 proving Theorem 1.2 about the case with pseudo-index $\iota_X > 1$.

In Section 3 we consider the following invariant of X:

$$c_X := \max\{\operatorname{codim} \mathcal{N}_1(D, X) \mid D \text{ is a prime divisor in } X\}.$$

In terms of this invariant, our main result is that $c_X \leq 8$, and if $c_X \geq 3$, then either X is a product, or $c_X = 3$ and X has a flat fibration onto an $(n-2)$-dimensional Fano manifold (see Theorem 3.3 for a precise statement). The proof of this result is quite long: it takes the whole Section 3, and is divided in several steps; see 3.5 for a plan. The strategy is to apply the construction of Section 2 to prime divisors of "minimal Picard number", *i.e.*, with $\operatorname{codim} \mathcal{N}_1(D,X) = c_X$. We show that there exists a prime divisor E_0 with $\operatorname{codim} \mathcal{N}_1(E_0,X) = c_X$, such that E_0 is a smooth $\mathbb{P}^1$-bundle with $E_0 \cdot f_0 = -1$, where $f_0 \subset E_0$ is a fiber. Applying the previous results to E_0, we obtain a bunch of disjoint divisors with a $\mathbb{P}^1$-bundle structure, and we use them to show that X is a product, or to construct a fibration in Del Pezzo surfaces.

Finally in Section 4 we use this result (Theorem 3.3) to prove the remaining results stated above: Theorem 1.1 and its Corollaries 1.3 to 1.6.

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Notation and terminology

We work over the field of complex numbers. A *manifold* is a smooth variety. A $\mathbb{P}^1$*-bundle* is a projectivization of a rank 2 vector bundle.

Let X be a projective variety.

$\mathcal{N}_1(X)$ (respectively, $\mathcal{N}^1(X)$) is the $\mathbb{R}$-vector space of one-cycles (respectively, Cartier divisors) with real coefficients, modulo numerical equivalence.

$[C]$ is the numerical equivalence class in $\mathcal{N}_1(X)$ of a curve $C \subset X$; $[D]$ is the numerical equivalence class in $\mathcal{N}^1(X)$ of a $\mathbb{Q}$-Cartier divisor D in X.

If $E \subset X$ is an irreducible closed subset and $C \subset E$ is a curve, $[C]_E$ is the numerical equivalence class of C in $\mathcal{N}_1(E)$.

The symbol $\equiv$ stands for numerical equivalence (for both 1-cycles and $\mathbb{Q}$-Cartier divisors).

For any $\mathbb{Q}$-Cartier divisor D in X, $D^{\perp} := \{\gamma \in \mathcal{N}_1(X) \mid D \cdot \gamma = 0\}$.

$\operatorname{NE}(X) \subset \mathcal{N}_1(X)$ is the convex cone generated by classes of effective curves, and $\overline{\operatorname{NE}}(X)$ is its closure.

An *extremal ray* R of X is a one-dimensional face of $\overline{\operatorname{NE}}(X)$; $\operatorname{Locus}(R) \subseteq X$ is the union of all curves whose class is in R.

If R is an extremal ray of X and D is a $\mathbb{Q}$-Cartier divisor in X, we say that $D \cdot R > 0$, respectively $D \cdot R = 0$, etc. if for $\gamma \in R \smallsetminus \{0\}$ we have $D \cdot \gamma > 0$, respectively $D \cdot \gamma = 0$, etc.

Assume that X is normal.

A *contraction* of X is a surjective morphism with connected fibers $\varphi \colon X \to Y$, where Y is normal and projective; $\operatorname{NE}(\varphi)$ is the face of $\overline{\operatorname{NE}}(X)$ generated by classes of curves contracted by φ.