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MONODROMY AND TOPOLOGICAL CLASSIFICATION OF GERMS OF HOLOMORPHIC FOLIATIONS

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Abstract. – We give a complete topological classification of germs of holomorphic foliations in the plane under rather generic conditions. The key point is the introduction of a new topological invariant called monodromy representation. This monodromy contains all the relevant dynamical information, in particular the projective holonomy representations whose topological invariance was conjectured in the eighties by Cerveau and Sad and is proved here under mild hypotheses.

Résumé. – Nous donnons une classification topologique complète des germes de feuilletages holomorphes dans le plan, sous des conditions de type plutôt générique. Le point-clé est l'introduction d'un nouvel invariant topologique appelé représentation de monodromie. Cette monodromie contient toutes les informations dynamiques pertinentes, en particulier les représentations d'holonomie projective dont l'invariance topologique a été conjecturée dans les années quatre-vingt par Cerveau et Sad et est prouvée ici sous des hypothèses faibles.

1. Introduction

The objective of this paper is to provide a complete topological classification of germs of singular non-dicritical holomorphic foliations $\mathcal{F}$ at $(0,0) \in \mathbb{C}^2$ under very generic conditions. To do this we introduce a new topological invariant which is a representation of the fundamental group of the complement of the separatrix curve into a suitable automorphism group. We shall call this representation the *monodromy of the foliation germ*.

In fact, the motivation of this work was the following conjecture of D. Cerveau and P. Sad in 1986, cf. [3, page 246]. Consider two germs of foliations defined by germs of differential holomorphic 1-forms ω and ω' at $(0,0) \in \mathbb{C}^2$.

Conjecture 1.1 (Cerveau-Sad). – *If ω and ω' are topologically conjugate and if ω is a generalized curve, then their respective projective holonomy representations are conjugate.*

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It was given in two forms, each of them with natural generic hypothesis concerning the germ of the foliation $\underline{\mathcal{F}}$ along the *exceptional divisor* $\mathcal{E}_{\mathcal{F}} := E_{\mathcal{F}}^{-1}(0)$ of the reduction $E_{\mathcal{F}} : \mathcal{B}_{\mathcal{F}} \to \mathbb{C}^2$ of the singularity of $\mathcal{F}$, cf. [18, 12]. The weak form (named Conjecture A) assumes that the separatrix curve is the union of smooth and transverse branches. In particular, $E_{\mathcal{F}}$ corresponds to a single blow-up. The strong form (named Conjecture B) only asks that the reduced foliation $\underline{\mathcal{F}} = E_{\mathcal{F}}^*(\mathcal{F})$ on $\mathcal{B}_{\mathcal{F}}$ does not have any saddle-node singularity.

Conjecture A was established by one of us in [8]. We give here an affirmative answer to Conjecture B. More precisely, Theorem I below gives a list of topological invariants containing the projective holonomy representations. In turn Theorem II gives a complete topological classification.

It is worthwhile to stress here that through the whole paper all the topological conjugations between foliations that we consider are supposed to preserve the orientations of the ambient space and also the leaves orientations.

As in the situation considered by D. Cerveau and P. Sad, we restrict our attention to a reasonable class of foliations that are going to be called Generic General Type. Let $\mathcal{F}$ be a non-dicritical foliation, i.e., having a finite number n of irreducible analytic germ curves $S_1, \ldots, S_n$ invariant by $\mathcal{F}$, which are called *separatrices*. Here it is worth to recall the celebrated Separatrix Theorem of [2] asserting that $n > 0$. In the sequel we will call $S_{\mathcal{F}} := \bigcup_{i=1}^n S_i$ the *separatrix curve* of $\mathcal{F}$. Following [9] we say that the foliation $\mathcal{F}$ is of *General Type* if all the singularities of $\underline{\mathcal{F}}$ which are not linearizable are resonant, more precisely:

(GT) *for each singularity of $\underline{\mathcal{F}}$ there are local holomorphic coordinates (u, v) such that $\underline{\mathcal{F}}$ is locally defined by a holomorphic 1-form of one of the two following types:*
  (i) $\lambda_1 u dv + \lambda_2 v du$*, with* $\lambda_1\lambda_2 \neq 0$ *and* $\lambda_1/\lambda_2 \notin \mathbb{Q}_{<0}$ *(linearizable singularity),*
  (ii) $(\lambda_1 u + \cdots)dv + (\lambda_2 v + \cdots)du$*, with* $\lambda_1, \lambda_2 \in \mathbb{N}^*$ *(resonant saddle).*

To introduce the additional genericity condition (G) we recall first that a singularity of $\underline{\mathcal{F}}$ is of *nodal type* if it can be locally written as

$$(\lambda_1 u + \cdots)dv + (\lambda_2 v + \cdots)du,$$

with $\lambda_1\lambda_2 \neq 0$ and $\lambda_1/\lambda_2 \in \mathbb{R}_{<0} \setminus \mathbb{Q}_{<0}$. Such singularities are always linearizable and consequently the only local analytical invariant of a node is its Camacho-Sad index $-\lambda_1/\lambda_2$. The topological specificity of a nodal singularity s is the existence, in any small neighborhood of s, of a saturated closed set whose complement is an open disconnected neighborhood of the two punctured local separatrices of the node. We call *nodal separator* such a saturated closed set. We denote by $\mathrm{Node}(\underline{\mathcal{F}})$ the set of nodal singularities of $\underline{\mathcal{F}}$. With this notation the genericity condition can be stated as follows:

(G) *The closure of each connected component of* $\mathcal{E}_{\mathcal{F}} \setminus (\mathrm{Node}(\underline{\mathcal{F}}) \cap \mathrm{Sing}(\mathcal{E}_{\mathcal{F}}))$ *contains an irreducible component of the exceptional divisor* $\mathcal{E}_{\mathcal{F}}$ *having a non solvable holonomy group.*

Notice that when $\mathrm{Node}(\underline{\mathcal{F}}) = \varnothing$, the genericity condition (G) only asks for a single irreducible component of $\mathcal{E}_{\mathcal{F}}$ having a non solvable holonomy group. In the space of coefficients of the germ of holomorphic 1-form defining the foliation this condition is generic in the sense of the Krull topology, cf. [6].

A foliation satisfying Conditions (G) and (GT) above will be called *Generic General Type*. For such a foliation $\mathcal{F}$, if $\mathrm{Node}(\underline{\mathcal{F}}) = \varnothing$ Theorem I below provides a list of topological invariants. In the case that $\mathrm{Node}(\underline{\mathcal{F}}) \neq \varnothing$ we must restrict the class of topological conjugations in order to keep their invariance. In fact, the first version [11] of this work dealt only with Generic General Type foliations $\mathcal{F}$ satisfying the additional requirement $\mathrm{Node}(\underline{\mathcal{F}}) = \varnothing$. Here this hypothesis is eliminated by modifying slightly the statements and the proofs given in [11]. In practice, this is done by adding a prefix "$\mathcal{N}$-" to some notions whose new meaning is made precise when they appear for the first time. We recommend the reader to ignore all the prefix "$\mathcal{N}$-" in a first reading.

Definitions for the nodal case

- A *nodal separatrix of* $\mathcal{F}$ is a separatrix whose strict transform by the reduction map $E_{\mathcal{F}}$ meets the exceptional divisor at a nodal singular point of $\underline{\mathcal{F}}$.
- An $\mathcal{N}$*-separator of* $\underline{\mathcal{F}}$ is the union of a system of nodal separators, one for each point in $\mathrm{Node}(\underline{\mathcal{F}}) \cap \mathrm{Sing}(\mathcal{E}_{\mathcal{F}})$ jointly with some tubular neighborhoods of the strict transforms of the nodal separatrices of $\mathcal{F}$. An $\mathcal{N}$*-separator of* $\mathcal{F}$ is the image by $E_{\mathcal{F}}$ of an $\mathcal{N}$-separator of $\underline{\mathcal{F}}$. If $\mathrm{Node}(\underline{\mathcal{F}}) = \varnothing$, an $\mathcal{N}$-separator is the empty set.
- An $\mathcal{N}$*-topological conjugation between two foliation germs* $\mathcal{F}$ *and* $\mathcal{F}'$ is a germ of homeomorphism h preserving the orientation of the ambient space as well as the orientation of the leaves, which is a topological conjugation between $\mathcal{F}$ and $\mathcal{F}'$, such that for each nodal separatrix S_j of $\mathcal{F}$, $h(S_j)$ is a nodal separatrix of $\mathcal{F}'$ and the Camacho-Sad indices of $\underline{\mathcal{F}}$ and $\underline{\mathcal{F}'}$ along the strict transforms of S_j and $h(S_j)$ coincide.
- An $\mathcal{N}$*-transversely holomorphic conjugation* between $\mathcal{F}$ and $\mathcal{F}'$ (resp. $\underline{\mathcal{F}}$ and $\underline{\mathcal{F}}'$) is an $\mathcal{N}$-topologically conjugation between these foliations, which is transversely holomorphic on the complementary of some $\mathcal{N}$-separator of $\mathcal{F}$ (resp. $\underline{\mathcal{F}}$).

Clearly the notions of $\mathcal{N}$-topological conjugation and $\mathcal{N}$-transversely holomorphic conjugation coincide with the usual notions of topological conjugation and transversely holomorphic conjugation, when $\mathrm{Node}(\underline{\mathcal{F}}) = \varnothing$. In Section 7.2, cf. Remark 7.2.1, we shall prove that:

- *Any topological conjugation which is transversely holomorphic in a neighborhood of each nodal separatrix minus the origin is an* $\mathcal{N}$*-topological conjugation.*

In particular, any transversely holomorphic conjugation is an $\mathcal{N}$-topological conjugation(1). In order to assure the transverse holomorphy of a conjugation we shall use a generalized form of the following theorem of J. Rebelo [16]:

Theorem 1.2 (Transverse Rigidity Theorem). – *Every topological conjugation between two germs of non-dicritical holomorphic foliations satisfying the genericity condition* (G) *and having singularities, after reduction, of type* $(\lambda_1 u + \cdots)dv + (\lambda_2 v + \cdots)du$ *with* $\lambda_1\lambda_2 \neq 0$, $\lambda_1/\lambda_2 \notin \mathbb{R}_{<0}$, *is transversely conformal.*

(1) Added in proof: R. Rosas has shown in a recent preprint [17] that every topological conjugation is an $\mathcal{N}$-topological conjugation.

In fact the proof provided in [16] shows that if we allow nodal singularities then each connected component of $\mathcal{E}_{\mathcal{F}} \setminus (\mathrm{Node}(\underline{\mathcal{F}}) \cap \mathrm{Sing}(\mathcal{E}_{\mathcal{F}}))$ possesses an open neighborhood W such that the restriction of the topological conjugation to $E_{\mathcal{F}}(W) \setminus \{0\}$ is transversely conformal. The extended version of the Transverse Rigidity Theorem asserts:

(TRT) *For any orientation preserving topological conjugation Φ between two germs of non-dicritical generalized curves satisfying condition (G) we have that Φ is an $\mathcal{N}$-topological conjugation if and only if Φ is an $\mathcal{N}$-transversely holomorphic conjugation.*

Theorem 1.3 (Theorem I). – *For every non-dicritical Generic General Type foliation $\mathcal{F}$, the analytic type of the projective holonomy representation of each irreducible component of the exceptional divisor $\mathcal{E}_{\mathcal{F}}$ is a topological invariant when* $\mathrm{Node}(\underline{\mathcal{F}}) = \varnothing$. *More generally, the semilocal data $\mathcal{SL}(\mathcal{F})$ constituted by*

- *the topological type of the embedding of the total separatrix curve $S_{\mathcal{F}}$ of $\mathcal{F}$ into $(\mathbb{C}^2, 0)$,*
- *the collection of local analytic types $[\underline{\mathcal{F}}_s]^{\mathrm{hol}}$ of the reduced foliation $\underline{\mathcal{F}}$ at each singular point $s \in \mathrm{Sing}(\underline{\mathcal{F}})$, codifying in particular the Camacho-Sad index $\mathrm{CS}(\underline{\mathcal{F}}, D, s)$ of $\underline{\mathcal{F}}$ at every singular point s along each irreducible component D of $\mathcal{E}_{\mathcal{F}}$ containing s,*
- *the analytic type of the holonomy representation $\mathcal{H}_{\underline{\mathcal{F}}, D}$ of each irreducible component D of $\mathcal{E}_{\mathcal{F}}$,*

is an $\mathcal{N}$-topological invariant [(2)] *of the germ of $\mathcal{F}$ at $0 \in \mathbb{C}^2$.*

Notice that the Camacho-Sad index $\mathrm{CS}(\underline{\mathcal{F}}, D, s)$ determines the analytic type of $\underline{\mathcal{F}}$ at s when s is not a resonant singularity after the assumption (GT). On the other hand, the genericity condition (G) is strictly necessary in Theorem I. Indeed, inside the family of homeomorphisms $\Psi(x, y) = (x|x|^a, y|y|^b)$ there is a topological conjugation between any pair of linear hyperbolic singularities having different Camacho-Sad indices.

Theorem I asserts that $\mathcal{SL}(\mathcal{F})$ is a topological invariant for the class of Generic General Type foliations with $\mathrm{Node}(\underline{\mathcal{F}}) = \varnothing$. In fact, the equality $\mathcal{SL}(\mathcal{F}) = \mathcal{SL}(\mathcal{F}')$ needs to be specified because the index sets of the families can be different for $\mathcal{F}$ and $\mathcal{F}'$. In order to do this, we recall that a topological conjugation between $\mathcal{F}$ and $\mathcal{F}'$ as above transforms $S_{\mathcal{F}}$ into $S_{\mathcal{F}'}$ and induces a unique homeomorphism

$$\Psi^\sharp : \mathcal{E}_{\mathcal{F}} \to \mathcal{E}_{\mathcal{F}'}, \quad \Psi^\sharp(\mathrm{Sing}(\underline{\mathcal{F}})) = \mathrm{Sing}(\underline{\mathcal{F}}')$$

between the exceptional divisors up to isotopy. This is a consequence of the following result proved in a previous work [10].

Theorem 1.4 (Marking Theorem). – *Let S and S' be two germs of analytic curves at the origin in $\mathbb{C}^2$ and let $h : (\mathbb{C}^2, 0) \xrightarrow{\sim} (\mathbb{C}^2, 0)$ be a germ of homeomorphism such that $h(S) = S'$. If E_S and $E_{S'}$ denote the minimal reduction of singularities of S and S', then there is a germ of homeomorphism $h_1 : (\mathbb{C}^2, 0) \xrightarrow{\sim} (\mathbb{C}^2, 0)$ such that:*

(i) *$h_1(S) = S'$ and the restrictions of h and h_1 to the complements of S and S' are homotopic,*

(2) Added in proof: By applying the previously cited preprint of R. Rosas [17], Theorem I can be rephrased as "$\mathcal{SL}(\mathcal{F})$ is a topological invariant".