ISSN 0012-9593

*quatrième série - tome 45* *fascicule 3* *mai-juin 2012*

# *ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE*

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SOCIÉTÉ MATHÉMATIQUE DE FRANCE

*Ann. Scient. Éc. Norm. Sup.*
4e série, t. 45, 2012, p. 447 à 489

# HOLOMORPHIC ACTIONS, KUMMER EXAMPLES, AND ZIMMER PROGRAM

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Abstract. – We classify compact Kähler manifolds $M$ of dimension $n \geq 3$ on which acts a lattice of an almost simple real Lie group of rank $\geq n-1$. This provides a new line in the so-called Zimmer program, and characterizes certain complex tori as compact Kähler manifolds with large automorphisms groups.

Résumé. – Nous classons les variétés compactes kählériennes $M$ de dimension $n \geq 3$ munies d'une action d'un réseau $\Gamma$ dans un groupe de Lie réel presque simple de rang $\geq n-1$. Ceci complète le programme de Zimmer dans ce cadre, et caractérise certains tores complexes compacts par des propriétés de leur groupe d'automorphismes.

## 1. Introduction

### 1.1. Zimmer Program

Let $G$ be an almost simple real Lie group. The *real rank* $\mathsf{rk}_{\mathbf{R}}(G)$ of $G$ is the dimension of a maximal Abelian subgroup of $G$ that acts by $\mathbf{R}$-diagonalizable endomorphisms in the adjoint representation of $G$ on its Lie algebra $\mathfrak{g}$; for example, the real Lie groups $\mathsf{SL}_n(\mathbf{R})$ and $\mathsf{SL}_n(\mathbf{C})$ have rank $n-1$. When $\mathsf{rk}_{\mathbf{R}}(G)$ is at least 2, we say that $G$ is a *higher rank* almost simple Lie group. Let $\Gamma$ be a *lattice* in a higher rank Lie group $G$; by definition, $\Gamma$ is a discrete subgroup of $G$ such that $G/\Gamma$ has finite Haar volume. Margulis superrigidity theorem implies that all finite dimensional linear representations of $\Gamma$ are built from representations in unitary groups and representations of the Lie group $G$ itself. In particular, there is no faithful linear representation of $\Gamma$ in dimension $\leq \mathsf{rk}_{\mathbf{R}}(G)$ (see Remark 3.4 below).

Zimmer's program predicts a similar picture for actions of $\Gamma$ by diffeomorphisms on compact manifolds, at least when the dimension $\dim(V)$ of the manifold $V$ is close to the minimal dimension of non trivial linear representations of $G$ (see [27]). For instance, a central conjecture asserts that lattices in simple Lie groups of rank $n$ do not act faithfully on compact manifolds of dimension less than $n$ (see [64, 63, 65, 31]).

In this article, we pursue the study of Zimmer's program for holomorphic actions on compact Kähler manifolds, as initiated in [16] and [19, 20].

### 1.2. Automorphisms

Let $M$ be a compact complex manifold of dimension $n$. By definition, diffeomorphisms of $M$ which are holomorphic are called *automorphisms*. According to Bochner and Montgomery [11, 14], the group $\mathsf{Aut}(M)$ of all automorphisms of $M$ is a complex Lie group, its Lie algebra is the algebra of holomorphic vector fields on $M$. Let $\mathsf{Aut}(M)^0$ be the connected component of the identity in $\mathsf{Aut}(M)$, and

$$\mathsf{Aut}(M)^\sharp = \mathsf{Aut}(M)/\mathsf{Aut}(M)^0$$

be the group of connected components. This group can be infinite, and is hard to describe: For example, it is not known whether there exists a compact complex manifold $M$ for which $\mathsf{Aut}(M)^\sharp$ is not finitely generated.

When $M$ is a Kähler manifold, Lieberman and Fujiki proved that $\mathsf{Aut}(M)^0$ has finite index in the kernel of the action of $\mathsf{Aut}(M)$ on the cohomology of $M$ (see [28, 46]). Thus, if a subgroup $\Gamma$ of $\mathsf{Aut}(M)$ embeds into $\mathsf{Aut}(M)^\sharp$, the action of $\Gamma$ on the cohomology of $M$ has finite kernel; in particular, the group $\mathsf{Aut}(M)^\sharp$ almost embeds in the group $\mathrm{Mod}(M)$ of isotopy classes of smooth diffeomorphisms of $M$. When $M$ is simply connected or, more generally, has nilpotent fundamental group, $\mathrm{Mod}(M)$ is naturally described as the group of integer matrices in a linear algebraic group (see [56]). Thus, $\mathsf{Aut}(M)^\sharp$ sits naturally in an arithmetic lattice. Our main result goes in the other direction: It describes the largest possible lattices contained in $\mathsf{Aut}(M)^\sharp$.

### 1.3. Rigidity and Kummer examples

The main examples that provide large groups $\Gamma \subset \mathsf{Aut}(M)^\sharp$ come from linear actions on carefully chosen complex tori.

Example 1.1. – Let $E = \mathbf{C}/\Lambda$ be an elliptic curve and $n$ be a positive integer. Let $T$ be the torus $E^n = \mathbf{C}^n/\Lambda^n$. The group $\mathsf{Aut}(T)$ is the semi-direct product of $\mathsf{SL}(n, \mathsf{End}(E))$ by $T$, acting by translations on itself. In particular, the connected component $\mathsf{Aut}(T)^0$ coincides with the group of translations, and $\mathsf{Aut}(T)$ contains all linear transformations $z \mapsto B(z)$ where $B$ is in $\mathsf{SL}_n(\mathbf{Z})$. If $\Lambda$ is the lattice of integers $\mathcal{O}_d$ in an imaginary quadratic number field $\mathbf{Q}(\sqrt{d})$, where $d$ is a squarefree negative integer, then $\mathsf{Aut}(T)$ contains a copy of $\mathsf{SL}_n(\mathcal{O}_d)$.

Example 1.2. – Starting with the previous example, one can change $\Gamma$ into a finite index subgroup $\Gamma_0$, and change $T$ into a quotient $T/F$ where $F$ is a finite subgroup of $\mathsf{Aut}(T)$ which is normalized by $\Gamma_0$. In general, $T/F$ is an orbifold (a compact manifold with quotient singularities), and one needs to resolve the singularities in order to get an action on a smooth manifold $M$. The second operation that can be done is blowing up finite orbits of $\Gamma$. This provides infinitely many compact Kähler manifolds of dimension $n$ with actions of lattices $\Gamma \subset \mathsf{SL}_n(\mathbf{R})$ (resp. $\Gamma \subset \mathsf{SL}_n(\mathbf{C})$).

In these examples, the group $\Gamma$ is a lattice in a real Lie group of rank $(n-1)$, namely $\mathsf{SL}_n(\mathbf{R})$ or $\mathsf{SL}_n(\mathbf{C})$, and $\Gamma$ acts on a manifold $M$ of dimension $n$. Moreover, the action of $\Gamma$ on the cohomology of $M$ has finite kernel and a finite index subgroup of $\Gamma$ embeds in $\mathsf{Aut}(M)^\sharp$. Since this kind of construction is at the heart of the article, we introduce the following definition, which is taken from [17, 19].

DEFINITION 1.3. – *Let $\Gamma$ be a group, and $\rho : \Gamma \to \mathsf{Aut}(M)$ a morphism into the group of automorphisms of a compact complex manifold $M$. This morphism is a* Kummer example *(or, equivalently, is of* Kummer type*) if there exists*

- *a birational morphism $\pi : M \to M_0$ onto an orbifold $M_0$,*
- *a finite orbifold cover $\epsilon : T \to M_0$ of $M_0$ by a torus $T$, and*
- *morphisms $\eta : \Gamma \to \mathsf{Aut}(T)$ and $\eta_0 : \Gamma \to \mathsf{Aut}(M_0)$*

*such that $\epsilon \circ \eta(\gamma) = \eta_0(\gamma) \circ \epsilon$ and $\eta_0(\gamma) \circ \pi = \pi \circ \rho(\gamma)$ for all $\gamma$ in $\Gamma$.*

The notion of *orbifold* used in this text refers to compact complex analytic spaces with a finite number of singularities of quotient type; in other words, $M_0$ is locally the quotient of $(\mathbf{C}^n, 0)$ by a finite group of linear transformations (see Section 2.3).

Since automorphisms of a torus $\mathbf{C}^n/\Lambda$ are covered by affine transformations of $\mathbf{C}^n$, all Kummer examples come from actions of affine transformations on affine spaces.

### 1.4. Results

The following statement is our main theorem. It confirms Zimmer's program, in its strongest versions, for holomorphic actions on compact Kähler manifolds: We get a precise description of all possible actions of lattices $\Gamma \subset G$ for $\mathsf{rk}_{\mathbf{R}}(G) \geq \dim_{\mathbf{C}}(M) - 1$.

MAIN THEOREM. – *Let $G$ be an almost simple real Lie group and $\Gamma$ be a lattice in $G$. Let $M$ be a compact Kähler manifold of dimension $n \geq 3$. Let $\rho : \Gamma \to \mathsf{Aut}(M)$ be a morphism with infinite image.*

(0) *The real rank $\mathsf{rk}_{\mathbf{R}}(G)$ is at most equal to the complex dimension of $M$.*

(1) *If $\mathsf{rk}_{\mathbf{R}}(G) = \dim(M)$, the group $G$ is locally isomorphic to $\mathsf{SL}_{n+1}(\mathbf{R})$ or $\mathsf{SL}_{n+1}(\mathbf{C})$ and $M$ is biholomorphic to the projective space $\mathbb{P}^n(\mathbf{C})$.*

(2) *If $\mathsf{rk}_{\mathbf{R}}(G) = \dim(M) - 1$, there exists a finite index subgroup $\Gamma_0$ in $\Gamma$ such that either*

(2-a) *$\rho(\Gamma_0)$ is contained in $\mathsf{Aut}(M)^0$, and $\mathsf{Aut}(M)^0$ contains a subgroup which is locally isomorphic to $G$, or*

(2-b) *$G$ is locally isomorphic to $\mathsf{SL}_n(\mathbf{R})$ or $\mathsf{SL}_n(\mathbf{C})$, and the morphism $\rho : \Gamma_0 \to \mathsf{Aut}(M)$ is a Kummer example.*

Moreover, all examples corresponding to assertion (2-a) are described in Section 4.6 and all Kummer examples of assertion (2-b) are described in Section 7. In particular, for these Kummer examples, the complex torus $T$ associated to $M$ and the lattice $\Gamma$ fall in one of the following three possible examples:

– $\Gamma \subset \mathsf{SL}_n(\mathbf{R})$ is commensurable to $\mathsf{SL}_n(\mathbf{Z})$ and $T$ is isogenous to the product of $n$ copies of an elliptic curve $\mathbf{C}/\Lambda$;

– $\Gamma \subset \mathsf{SL}_n(\mathbf{C})$ is commensurable to $\mathsf{SL}_n(\mathcal{O}_d)$ where $\mathcal{O}_d$ is the ring of integers in $\mathbf{Q}(\sqrt{d})$ for some negative integer $d$, and $T$ is isogenous to the product of $n$ copies of the elliptic curve $\mathbf{C}/\mathcal{O}_d$;

– In the third example, $n = 2k$ is even. There are positive integers $a$ and $b$ such that the quaternion algebra $\mathbf{H}_{a,b}$ over the rational numbers $\mathbf{Q}$ defined by the basis $(1, \mathsf{i}, \mathsf{j}, \mathsf{k})$, with

$$\mathsf{i}^2 = a,\ \mathsf{j}^2 = b,\ \mathsf{ij} = \mathsf{k} = -\mathsf{ji}$$

is an indefinite quaternion algebra and the lattice $\Gamma$ is commensurable to the lattice $\mathsf{SL}_k(H_{a,b}(\mathbf{Z}))$. The torus $T$ is isogenous to the product of $k$ copies of an Abelian surface $Y$ which contains $\mathbf{H}_{a,b}(\mathbf{Q})$ in its endomorphism algebra $\mathsf{End}_{\mathbf{Q}}(Y)$. Once $a$ and $b$ are fixed, those surfaces $Y$ depend on one complex parameter, hence $T$ depends also on one parameter; for some parameters $Y$ is isogenous to the product of 2 copies of the elliptic curve $\mathbf{C}/\mathcal{O}_d$, and $T$ is isogenous to $(\mathbf{C}/\mathcal{O}_d)^n$ with $d = -ab$ (see §7 for precise definitions and details).

As a consequence, $\Gamma$ *is not cocompact, $T$ is an Abelian variety and $M$ is projective.*

Remark 1.4. – In dimension 2, [18] shows that all faithful actions of infinite discrete groups with Kazhdan property (T) by birational transformations on projective surfaces are birationally conjugate to actions by automorphisms on the projective plane $\mathbb{P}^2(\mathbf{C})$ ; thus, part (1) of the Main Theorem holds in the more general setting of birational actions and groups with Kazhdan property (T). Part (2) does not hold in dimension 2 for lattices in the rank 1 Lie group $\mathsf{SO}_{1,n}(\mathbf{R})$ (see [18, 26] for examples).

### 1.5. Strategy of the proof and complements

Sections 2 and 3 contain important preliminary facts, as well as a side result which shows how representation theory and Hodge theory can be used together in our setting (see §3.4).

The proof of the Main Theorem starts in §4: Assertion (1) is proved, and a complete list of all possible pairs $(M, G)$ that appear in assertion (2-a) is obtained. This makes use of a previous result on Zimmer conjectures in the holomorphic setting (see [16], in which assertion (0) is proved), and classification of homogeneous or quasi-homogeneous spaces (see [2, 32, 40]). On our way, we describe $\Gamma$-invariant analytic subsets $Y \subset M$ and show that these subsets can be blown down to quotient singularities.

The core of the paper is to prove assertion (2-b) when the image $\rho(\Gamma_0)$ is not contained in $\mathsf{Aut}(M)^0$ (for all finite index subgroups of $\Gamma$) and $\mathsf{rk}_{\mathbf{R}}(G)$ is equal to $\dim(M) - 1$.

In that case, $\Gamma$ acts almost faithfully on the cohomology of $M$, and this linear representation extends to a continuous representation of $G$ on $H^*(M, \mathbf{R})$ (see §3). Section 5 shows that $G$ preserves a non-trivial cone contained in the closure of the Kähler cone $\mathcal{K}(M) \subset H^{1,1}(M, \mathbf{R})$; this general fact holds for all linear representations of semi-simple Lie groups $G$ for which a lattice $\Gamma \subset G$ preserves a salient cone. Section 5 can be skipped in a first reading.

Then, in §6, we apply ideas of Dinh and Sibony, of Zhang, and of our previous manuscripts together with representation theory. We fix a maximal torus $A$ in $G$ and study the eigenvectors of $A$ in the $G$-invariant cone: Hodge index Theorem constrains the set of weights and eigenvectors; since the Chern classes are invariant under the action of $G$, this provides strong constraints on them. When there is no $\Gamma$-invariant analytic subset of positive dimension, Yau's Theorem can then be used to prove that $M$ is a torus. To conclude the proof, we blow down all invariant analytic subsets to quotient singularities (see §4), and apply Hodge and Yau's Theorems in the orbifold setting.