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RELAXATION OF THE INCOMPRESSIBLE POROUS MEDIA EQUATION

BY LÁSZLÓ SZÉKELYHIDI JR.

ABSTRACT. – It was shown recently by Córdoba, Faraco and Gancedo in [1] that the 2D porous media equation admits weak solutions with compact support in time. The proof, based on the convex integration framework developed for the incompressible Euler equations in [4], uses ideas from the theory of laminates, in particular $T4$ configurations. In this note we calculate the explicit relaxation of IPM, thus avoiding $T4$ configurations. We then use this to construct weak solutions to the unstable interface problem (the Muskat problem), as a byproduct shedding new light on the gradient flow approach introduced by Otto in [12].

RÉSUMÉ. – Il a récemment été démontré par Córdoba, Faraco et Gancedo dans [1], que l'équation des milieux poreux en dimension 2 admet des solutions faibles avec support compact dans le temps. La démonstration, qui fait appel à la méthode par intégration convexe telle qu'elle a été développée dans [4], dans le contexte des équations d'Euler incompressibles, utilise certaines idées provenant de la théorie des « laminates », et en particulier les configurations dites $T4$. Dans cette note, nous calculons explicitement la relaxation du « IPM », évitant ainsi les configurations $T4$. Ceci nous permet ensuite de construire des solutions faibles au problème des interfaces instables (problème de Muskat) et a pour autre conséquence de clarifier l'approche par flot de gradient, introduite par Otto dans [12].

1. Introduction

We consider the incompressible porous media equation (IPM) in a 2-dimensional bounded domain $\Omega \subset \mathbb{R}^2$. The flow is described in Eulerian coordinates by a velocity

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field $v(x,t)$ and a pressure $p(x,t)$ obeying the conservation of mass and the conservation of momentum in the form of Darcy's law:

$$\partial_t \rho + \operatorname{div}(\rho v) = 0, \tag{IPM1}$$

$$\operatorname{div} v = 0, \tag{IPM2}$$

$$v + \nabla p = -(0, \rho). \tag{IPM3}$$

Here we chose x_1 as the horizontal and x_2 as the vertical direction, with the gravitational constant normalized to be 1. Equation (IPM2) amounts to the flow being incompressible, and this is coupled with the assumption that there is no flux across the boundary $\partial\Omega$, i.e., $v \cdot \nu = 0$ on $\partial\Omega$.

The system (IPM1)-(IPM3) can be used to model the flow of two immiscible fluids of different densities in a porous medium, or, equivalently, in a Hele-Shaw cell [15]. If initially the two fluids form a horizontal interface, with the heavier fluid on top, it is well known that the initial value problem, known as the Muskat problem, is ill-posed in classical function spaces [19, 17, 2]. Although some explicit solutions are known [7], there is no general existence theory, neither for the evolution problem for the interface, nor for weak solutions of IPM.

After a normalization we may assume that the density $\rho(x,t)$, indicating whether the pores at time t near location $x \in \Omega$ are filled with the lighter or the heavier fluid, takes the values ± 1. Hence, for the Muskat problem the Equations (IPM1)-(IPM3) should be complemented by

$$|\rho(x,t)| = 1 \quad \text{for a.e.} (x,t) \in \Omega \times (0,T). \tag{IPM4}$$

We remark in passing, that formally (IPM4) follows from (IPM1) if $|\rho(x,0)| = 1$ a.e., since the density ρ is simply transported by the flow. However, for weak solutions this transport property need not hold, as shown for instance by Theorem 1.2 below.

As usual, a weak solution to the system (IPM1)-(IPM3) with initial data $\rho_0 \in L^\infty(\Omega)$ is defined as a pair (ρ, v) with

$$\rho \in L^\infty(\Omega \times (0,T)), \quad v \in L^\infty(0,T; L^2(\Omega)),$$

such that for all $\phi \in C_c^\infty(\mathbb{R}^2 \times \mathbb{R})$ we have

$$\int_0^T \int_\Omega \rho(\partial_t \phi + v \cdot \nabla \phi)\, dx dt + \int_\Omega \rho_0(x)\phi(x,0)\, dx = 0, \tag{1}$$

$$\int_0^T \int_\Omega v \cdot \nabla \phi \, dx dt = 0, \tag{2}$$

and for all $\psi \in C_c^\infty(\Omega)$

$$\int_\Omega (v + (0,\rho)) \cdot \nabla^\perp \psi \, dx = 0. \tag{3}$$

We remark explicitly that (2) includes the no-flux boundary condition for v whereas in (3) the pressure p has been eliminated (observe also that there is no boundary condition on p).

Our main result can be stated as follows

THEOREM 1.1. – *Let $\Omega \subset \mathbb{R}^2$ be the unit square, and*

$$\rho_0(x) = \begin{cases} +1 & x_2 > 0, \\ -1 & x_2 < 0. \end{cases}$$

For any $T > 0$ there exist infinitely many weak solutions $\rho \in L^\infty(\Omega \times (0,T))$ of (IPM1)-(IPM4) *with initial data ρ_0.*

Recently, D. Córdoba, D. Faraco and F. Gancedo showed in [1], that on the 2-dimensional torus $\mathbb{T}^2$ the system (IPM1)-(IPM3) admits nontrivial weak solutions with compact support in time. More precisely

THEOREM 1.2 (Theorem 5.2, [1]). – *There exist infinitely many weak solutions to* (IPM1)-(IPM3) *with $(\rho, v) \in L^\infty(\mathbb{T}^2 \times \mathbb{R})$ such that*

$$|\rho(x,t)| = \begin{cases} 1 & \textit{a.e. } (x,t) \in \mathbb{T}^2 \times (0,T), \\ 0 & \textit{for } t < 0 \textit{ or } t > T. \end{cases}$$

There is a subtle but quite important difference between the solutions in Theorem 1.1 and Theorem 1.2. In the latter the initial data (in the sense of Equation (1)) is $\rho_0 = 0$, so that although (IPM4) holds for $t > 0$, it is not satisfied by ρ_0. An interpretation of this is that the fluid is in an infinitely mixed state at time $t = 0$ (cf. discussion in Section 4). In contrast, for Theorem 1.1 the fluid is not mixed at the initial time. As a consequence the solutions are forced to have finite mixing speed, an effect that cannot be seen in the solutions from Theorem 1.2. More precisely, the solutions in Theorem 1.1 all satisfy

$$\rho(x,t) = \begin{cases} +1 & x_2 > 2t, \\ -1 & x_2 < -2t. \end{cases} \tag{4}$$

Moreover, the solutions in Theorem 1.1 are in good agreement and show interesting connections to predictions concerning the coarse-grained density and the growth of the mixing zone made in [12, 13]. In [12] F. Otto introduced a relaxation approach for (IPM1)-(IPM4), in particular for the Muskat problem, based on a gradient flow formulation of IPM and using ideas from mass transport. It was shown that under certain assumptions there exists a *unique* "relaxed" solution $\overline{\rho}$, representing a kind of coarse-grained density. Moreover, Otto showed in [13, Remark 2.1] that, in general, the mixing zone (where the coarse-grained density $\overline{\rho}$ is strictly between ± 1) grows linearly in time as in (4), with the possible exception of a small set of volume fraction $O(t^{-1/2})$.

The proof of both Theorem 1.2 in [1] and Theorem 1.1 is based on the framework developed in [4] for the incompressible Euler equations, although there are several places where the authors in [1] had to modify the arguments. In technical terms, one of the crucial points in the general scheme of convex integration is to show that the relaxation with respect to the wave cone of a suitably defined constitutive set, the Λ-convex hull, contains the zero state in its interior. In [1] it was observed, that due to a lack of symmetry induced by the direction of gravity, this condition seems to fail for IPM; instead, a systematic method for obtaining a suitably modified constitutive set was introduced, based on so-called degenerate $T4$ configurations. The advantage of the method used in [1] is that it is rather robust, and can be used in situations where an explicit calculation of K^{Λ} is out of reach due to the high complexity (see also [9, 11]). Indeed, the same technique has been recently applied successfully to a large class of active scalar equations by R. Shvydkoy [16].

On the other hand, there are certain advantages to obtaining an explicit formula for the Λ-convex hull K^{Λ} rather than just showing that a fixed state is in the interior. The explicit formula allows one to identify "compatible boundary and initial conditions", for which the construction works. For the incompressible Euler equations, such initial conditions were called "wild initial data" in [5]. For IPM the explicit formula for the Λ-convex hull is necessary for studying the Muskat problem and leads to a concept of subsolution, analogously to Euler subsolutions in [4, 5]. In Section 4 we show that the relaxed solution $\overline{\rho}$ from [12] is very closely related to the concept of subsolution and in particular we construct weak solutions ρ_k such that $\rho_k \overset{*}{\rightharpoonup} \overline{\rho}$. The interpretation is that $\overline{\rho}$ is the coarse-grained density obtained from ρ_k. It is interesting to note in this connection, that, although weak solutions are clearly not unique, there is a way to identify a selection criterion among subsolutions which leads to uniqueness.

The paper is organized as follows. In Section 2 we recall from [1] how to reformulate (IPM1)-(IPM3) as a differential inclusion, and then we calculate explicitly the relaxation, more precisely the Λ-convex hull of the constitutive set. These computations form the main contribution of this paper. If one is only interested in weak solutions as defined in this introduction (where v can be *unbounded*), the "simpler" computations in Propositions 2.3 and 3.1 suffice. However, for completeness we include the computations that are required for *bounded* velocity v in Propositions 2.4 and 3.3.

Then, in Section 3 we show how the explicit form of the Λ-convex hull can be used in conjunction with the Baire category method to obtain weak solutions. For the convenience of the reader we include the details of the Baire category method in the appendix.

Finally, in Section 4 we use the framework to construct weak solutions to the unstable interface problem. In this section, Theorem 1.1 is restated and proved as Theorem 4.2. Moreover, we show in Proposition 4.3 that if the coarse-grained density is independent of the horizontal direction, the linear growth estimate of [13] is sharp, in the sense that there is no exceptional set. As a consequence, we can interpret the uniqueness result of Otto as selecting the subsolution with "maximal mixing". In this light it is of interest to note that the analogous criterion for the incompressible Euler equations would be "maximally dissipating" [3, 6, 5].

2. The relaxation of IPM

We start by setting

$$u := 2v + (0, \rho). \tag{5}$$

Then (IPM1)-(IPM3) can be rewritten as

$$\begin{aligned} \partial_t \rho + \operatorname{div} m &= 0, \\ \operatorname{div}(u - (0,\rho)) &= 0, \\ \operatorname{curl}(u + (0,\rho)) &= 0, \end{aligned} \tag{6}$$