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Christophe CAZANAVE

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ALGEBRAIC HOMOTOPY CLASSES OF RATIONAL FUNCTIONS

BY CHRISTOPHE CAZANAVE

ABSTRACT. – Let k be a field. We compute the set $\left[\mathbf{P}^1,\mathbf{P}^1\right]^{\mathrm{N}}$ of *naive* homotopy classes of pointed k-scheme endomorphisms of the projective line $\mathbf{P}^1$. Our result compares well with Morel's computation in [11] of the *group* $\left[\mathbf{P}^1,\mathbf{P}^1\right]^{\mathbf{A}^1}$ of $\mathbf{A}^1$-homotopy classes of pointed endomorphisms of $\mathbf{P}^1$: the set $\left[\mathbf{P}^1,\mathbf{P}^1\right]^{\mathrm{N}}$ admits an a priori monoid structure such that the canonical map $\left[\mathbf{P}^1,\mathbf{P}^1\right]^{\mathrm{N}} \to \left[\mathbf{P}^1,\mathbf{P}^1\right]^{\mathbf{A}^1}$ is a group completion.

RÉSUMÉ. – Soit k un corps. Nous déterminons l'ensemble $\left[\mathbf{P}^1,\mathbf{P}^1\right]^{\mathrm{N}}$ des classes d'homotopie *naïve* d'endomorphismes pointés de k-schémas de la droite projective $\mathbf{P}^1$. Notre résultat se compare bien avec le calcul de Morel [11] du *groupe* $\left[\mathbf{P}^1,\mathbf{P}^1\right]^{\mathbf{A}^1}$ des classes d'$\mathbf{A}^1$-homotopie d'endomorphismes pointés de $\mathbf{P}^1$: l'ensemble $\left[\mathbf{P}^1,\mathbf{P}^1\right]^{\mathrm{N}}$ admet *a priori* une structure de monoïde pour laquelle l'application canonique $\left[\mathbf{P}^1,\mathbf{P}^1\right]^{\mathrm{N}} \to \left[\mathbf{P}^1,\mathbf{P}^1\right]^{\mathbf{A}^1}$ est une complétion en groupe.

1. Introduction

The work of Fabien Morel and Vladimir Voevodsky on $\mathbf{A}^1$-homotopy theory [9, 12] provides a convenient framework to do algebraic topology in the setting of algebraic geometry. More precisely, for a fixed field k, Morel and Voevodsky defined an appropriate category of *spaces*, say $\mathcal{S}p$, containing the category of smooth algebraic k-varieties as a full subcategory, which they endowed with a suitable model structure, in the sense of Quillen's homotopical algebra [13]. Thus, given two spaces X and Y in $\mathcal{S}p$ (resp. two pointed spaces), the set $\{X,Y\}^{\mathbf{A}^1}$ (resp. the set $[X,Y]^{\mathbf{A}^1}$) of $\mathbf{A}^1$-homotopy classes of *unpointed* morphisms (resp. of *pointed* morphisms) from X to Y is well defined and has all the properties an algebraic topologist can expect. However, for concrete X and Y, these sets are in general hard to compute.

At the starting point of $\mathbf{A}^1$-homotopy theory is the notion of *naive homotopy*(1) between two morphisms in $\mathcal{S}p$. First introduced by Karoubi and Villamayor [6], this notion mimics

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(1) In [12], the authors use the terminology "elementary homotopy".

the usual one of homotopy between topological maps, replacing the unit interval $[0,1]$ by its algebraic analogue, the affine line $\mathbf{A}^1$.

DEFINITION 1.1. – Let X and Y be two spaces in $\mathcal{S}p$. A naive homotopy is a morphism in $\mathcal{S}p$

$$F : X \times \mathbf{A}^1 \longrightarrow Y.$$

The restriction $\sigma(F) := F_{|X\times\{0\}}$ is the source of the homotopy and $\tau(F) := F_{|X\times\{1\}}$ is its target. When X and Y have base points, say x_0 and y_0, we say that F is pointed if its restriction to $\{x_0\} \times \mathbf{A}^1$ is constant equal to y_0.

With this notion, one defines the set $\{X,Y\}^{\mathrm{N}}$ (resp. the set $[X,Y]^{\mathrm{N}}$) of unpointed (resp. of pointed) *naive* homotopy classes of morphisms from X to Y as the quotient of the set of unpointed (resp. of pointed) morphisms by the equivalence relation *generated* by unpointed (resp. by pointed) naive homotopies. These sets are sometimes easier to compute than their $\mathbf{A}^1$ analogues, but they are not very well behaved. There is a canonical comparison map

$$[X,Y]^{\mathrm{N}} \longrightarrow [X,Y]^{\mathbf{A}^1}$$

which in general is far from being a bijection. In this article, we study a particular situation where this map has a noteworthy behavior.

Let k be a base field. We focus on the set of pointed homotopy classes of k-scheme endomorphisms of the projective line $\mathbf{P}^1$ with base point $\infty := [1:0]$. The set $\left[\mathbf{P}^1,\mathbf{P}^1\right]^{\mathbf{A}^1}$ is computed by Fabien Morel in [11]. Note that, as the source space $\mathbf{P}^1$ is homotopy equivalent in $\mathcal{S}p$ to a suspension (see Lemma 3.20), the set $\left[\mathbf{P}^1,\mathbf{P}^1\right]^{\mathbf{A}^1}$ is endowed with a natural group structure, whose law is denoted by $\oplus^{\mathbf{A}^1}$.

On the other hand, by interpreting endomorphisms of $\mathbf{P}^1$ as rational functions, we define a monoid law $\oplus^{\mathrm{N}}$ on $\left[\mathbf{P}^1,\mathbf{P}^1\right]^{\mathrm{N}}$. Using this additional structure and a classical construction due to Bézout, we can give an explicit description of $\left[\mathbf{P}^1,\mathbf{P}^1\right]^{\mathrm{N}}$. Morel's computation combined with ours then leads to the following striking result.

THEOREM 1.2. – *The canonical map*

$$\left(\left[\mathbf{P}^1,\mathbf{P}^1\right]^{\mathrm{N}},\oplus^{\mathrm{N}}\right) \longrightarrow \left(\left[\mathbf{P}^1,\mathbf{P}^1\right]^{\mathbf{A}^1},\oplus^{\mathbf{A}^1}\right)$$

is a group completion.

Overview of the paper

Section 2 reviews the classical correspondence between scheme endomorphisms of the projective line and rational functions. This leads to a description of $\left[\mathbf{P}^1,\mathbf{P}^1\right]^{\mathrm{N}}$ as a set of algebraic homotopy classes of rational functions with coefficients in the field k.

Section 3 is the core of the article. In §3.1, we define a monoid structure on the scheme $\mathcal{F}$ of pointed rational functions. Through the correspondence of Section 2, it induces the monoid law $\oplus^{\mathrm{N}}$ which appears in Theorem 1.2. Then §3.2 reviews a classical construction due to Bézout, which associates to any rational function a non-degenerate symmetric k-bilinear form. We use it to define a homotopy invariant of rational functions taking values in some set of equivalence classes of symmetric k-bilinear forms. Our main result, stated in §3.3, shows that this invariant distinguishes exactly all the homotopy classes of rational functions. The proof is given in §3.4. Finally, §3.5 compares our result to the actual $\mathbf{A}^1$-homotopy classes of Morel, as in Theorem 1.2.

Section 4 discusses natural extensions of the previous computation. We first give a similar description of the set of *unpointed* naive homotopy classes of endomorphisms of $\mathbf{P}^1$ in §4.1. Next, in §4.2, we study the composition of endomorphisms of $\mathbf{P}^1$ in terms of our description of $[\mathbf{P}^1, \mathbf{P}^1]^{\mathrm{N}}$. Finally, in §4.3, we compute the set $[\mathbf{P}^1, \mathbf{P}^d]^{\mathrm{N}}$ of pointed naive homotopy classes of morphisms from $\mathbf{P}^1$ to $\mathbf{P}^d$ for every integer $d \geqslant 2$. Not surprisingly, this case is easier than the case $d = 1$. The result still compares well with Morel's computation of the actual $\mathbf{A}^1$-homotopy classes.

The article ends on an appendix proving the compatibility of the law $\oplus^{\mathrm{N}}$ on $[\mathbf{P}^1, \mathbf{P}^1]^{\mathrm{N}}$ with that $\oplus^{\mathbf{A}^1}$ on $[\mathbf{P}^1, \mathbf{P}^1]^{\mathbf{A}^1}$. This is a crucial part of the comparison of our results to those of Morel.

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The material presented here constitutes the first part of my PhD thesis [3]. The main result was first announced in the note [2] when $\mathrm{char}(k) \neq 2$. I am very much indebted to Jean Lannes for his precious and generous help. I am also grateful to the topology group of Bonn Universität for its hospitality while this article was written.

2. Rational functions and naive homotopies

We review the classical correspondence between pointed k-scheme endomorphisms of the projective line $(\mathbf{P}^1, \infty)$ and pointed rational functions with coefficients in k. Similarly, naive homotopies have a description in terms of pointed rational functions with coefficients in the ring $k[T]$.

Definition 2.1. – For an integer $n \geqslant 1$, the scheme $\mathscr{F}_n$ of pointed degree n rational functions is the open subscheme of the affine space $\mathbf{A}^{2n} = \operatorname{Spec} k[a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1}]$ complementary to the hypersurface of equation[(2)]

$$\mathrm{res}_{n,n}(X^n + a_{n-1}X^{n-1} + \cdots + a_0, b_{n-1}X^{n-1} + \cdots + b_0) = 0.$$

By convention, $\mathscr{F}_0 := \operatorname{Spec} k$.

Remark 2.2. – Let R be a k-algebra and n a non-negative integer. By the very definition, an R-point of $\mathscr{F}_n$ is a pair of polynomials $(A, B) \in R[X]^2$, where

- A is monic of degree n,
- B is of degree strictly less than n,
- the scalar $\mathrm{res}_{n,n}(A, B)$ is invertible in R.

Such an element is denoted by $\frac{A}{B}$ and is called a pointed degree n rational function with coefficients in R. In the sequel, it is useful to remark that given A and B as above, the

(2) The notation $\mathrm{res}_{n,n}(A, B)$ stands for the *resultant* of the polynomials A and B with conventions as in [1, §6, n° 6, IV].

condition $\mathrm{res}_{n,n}(A,B) \in R^\times$ is equivalent to the existence of a (necessarily unique) Bézout relation

$$AU + BV = 1$$

with U and V polynomials in $R[X]$ such that $\deg V \leqslant n-1$ (and $\deg U \leqslant n-2$ if $n \neq 0$).

Pointed k-scheme morphisms $\mathbf{P}^1 \to \mathbf{P}^1$ and pointed naive homotopies $F : \mathbf{P}^1 \times \mathbf{A}^1 = \mathbf{P}^1_{k[T]} \to \mathbf{P}^1$ are then described in terms of rational functions as follows.

Proposition 2.3. – *Let $R = k$ or $R = k[T]$. The datum of a pointed k-scheme morphism $f : \mathbf{P}^1_R \longrightarrow \mathbf{P}^1$ is equivalent to the datum of a non-negative integer n and of an element $\frac{A}{B} \in \mathcal{F}_n(R)$. The integer n is called the* degree *of f and is denoted* $\deg(f)$*; the scalar* $\mathrm{res}_{n,n}(A,B) \in R^\times = k^\times$ *is called the* resultant *of f and is denoted* $\mathrm{res}(f)$.

Proof. – This follows from the usual description of morphisms to a projective space (using the fact that the ring R is a UFD). □

Example 2.4. – Let n be a positive integer and b_0 be a unit in $k^\times$.

1. A polynomial $\frac{X^n + a_{n-1}X^{n-1} + \cdots + a_0}{b_0}$ is homotopic to its leading term $\frac{X^n}{b_0}$.
2. Let B be a polynomial of degree $\leqslant n-1$ such that $B(0) = b_0$. Then $\frac{X^n}{B} \overset{\mathrm{p}}{\sim} \frac{X^n}{b_0}$.

In general, given a random rational function, it is not a priori easy to find non-trivial homotopies. In Remark 3.2(2), we will indicate a way of producing some such homotopies.

Definition 2.5. – Let f and g be two pointed rational functions over k. We say that f and g are in the same pointed naive homotopy class, and we write $f \overset{\mathrm{p}}{\sim} g$, if there exists a finite sequence of pointed homotopies, say (F_i) with $0 \leqslant i \leqslant N$, such that

– $\sigma(F_0) = f$ and $\tau(F_N) = g$;
– for every $0 \leqslant i \leqslant N-1$, we have $\tau(F_i) = \sigma(F_{i+1})$.

The set of pointed naive homotopy classes $[\mathbf{P}^1, \mathbf{P}^1]^{\mathrm{N}}$ is thus the quotient set $\coprod_{n \geqslant 0} \mathcal{F}_n(k) /_{\overset{\mathrm{p}}{\sim}}$.

Note that Proposition 2.3 implies that two pointed rational functions which are in the same pointed naive homotopy class have same degree (and also same resultant). In particular, the set $[\mathbf{P}^1, \mathbf{P}^1]^{\mathrm{N}}$ splits into the disjoint union of its degreewise components

$$[\mathbf{P}^1, \mathbf{P}^1]^{\mathrm{N}} = \coprod_{n \geqslant 0} [\mathbf{P}^1, \mathbf{P}^1]^{\mathrm{N}}_n .$$

Remark 2.6. – It is convenient to reformulate the preceding discussion in terms of the "naive connected components" of the scheme of pointed rational functions.

For $\mathcal{G} : \mathrm{Alg}_k \longrightarrow \mathcal{S}et$ a functor from the category of k-algebras to that of sets, recall that $\pi_0^{\mathrm{N}}\mathcal{G} : \mathrm{Alg}_k \longrightarrow \mathcal{S}et$ is the functor which assigns to a k-algebra R the coequalizer of the double-arrow $\mathcal{G}(R[T]) \rightrightarrows \mathcal{G}(R)$ given by evaluation at $T = 0$ and $T = 1$. For every non-negative integer n, Proposition 2.3 gives a bijection

$$[\mathbf{P}^1, \mathbf{P}^1]^{\mathrm{N}}_n \simeq (\pi_0^{\mathrm{N}}\mathcal{F}_n)(k).$$

Note that by functoriality a k-scheme morphism $\phi : \mathcal{F}_n \longrightarrow \mathcal{X}$ induces a homotopy invariant $\pi_0^{\mathrm{N}}(\phi)(k) : (\pi_0^{\mathrm{N}}\mathcal{F}_n)(k) \longrightarrow (\pi_0^{\mathrm{N}}\mathcal{X})(k)$.