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Antoine TOUZÉ

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TROESCH COMPLEXES AND EXTENSIONS OF STRICT POLYNOMIAL FUNCTORS

BY ANTOINE TOUZÉ

ABSTRACT. – We develop a new approach of extension calculus in the category of strict polynomial functors, based on Troesch complexes. We obtain new short elementary proofs of numerous classical Ext-computations as well as new results.

In particular, we get a cohomological version of the "fundamental theorems" from classical invariant theory for GL_n for n big enough (and we give a conjecture for smaller values of n).

We also study the "twisting spectral sequence" $E^{s,t}(F,G,r)$ converging to the extension groups $\mathrm{Ext}^*_{\mathcal{P}_\Bbbk}(F^{(r)},G^{(r)})$ between the twisted functors $F^{(r)}$ and $G^{(r)}$. Many classical Ext computations simply amount to the collapsing of this spectral sequence at the second page (for lacunary reasons), and it is also a convenient tool to study the effect of the Frobenius twist on Ext groups. We prove many cases of collapsing, and we conjecture collapsing is a general fact.

RÉSUMÉ. – Nous développons une nouvelle approche des calculs d'Ext dans la catégorie des foncteurs strictement polynomiaux, en nous basant sur les complexes de Troesch. Nous obtenons ainsi des démonstrations élémentaires de nombreux calculs classiques et de nouveaux résultats.

En particulier, nous obtenons une version cohomologique des théorèmes fondamentaux de la théorie classique des invariants de GL_n pour n suffisamment grand (et nous donnons une conjecture pour les plus petites valeurs de n).

Nous étudions également une suite spectrale de torsion de Frobenius $E^{s,t}(F,G,r)$ qui converge vers les groupes d'extensions $\mathrm{Ext}^*_{\mathcal{P}_\Bbbk}(F^{(r)},G^{(r)})$ entre foncteurs précomposés par le twist de Frobenius. De nombreux calculs classiques équivalent à l'effondrement de cette suite spectrale à la seconde page (par lacunarité), et elle constitue également un outil pratique pour étudier l'effet de la torsion de Frobenius sur les groupes d'extensions. Nous démontrons de nombreux cas d'effondrement, et nous conjecturons que l'effondrement a toujours lieu.

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1. Introduction

Let $\Bbbk$ be a field of prime characteristic p. In [11], Friedlander and Suslin introduced the category $\mathcal{P}_{\Bbbk}$ of strict polynomial functors (of finite type) over $\Bbbk$. Let $\mathcal{V}_{\Bbbk}$ be the category of finite dimensional $\Bbbk$-vector spaces. Roughly speaking, objects of $\mathcal{P}_{\Bbbk}$ are functors $F : \mathcal{V}_{\Bbbk} \to \mathcal{V}_{\Bbbk}$ with some additional polynomial structure, so that the following property holds. If G is an algebraic group (or group scheme) over $\Bbbk$ acting rationally on V, then functoriality defines a *rational* action of G on $F(V)$. Such functors occur very frequently in representation theory. Typical examples are symmetric powers S^d, tensor powers $\otimes^d$, exterior powers Λ^d or divided powers Γ^d.

The category $\mathcal{P}_{\Bbbk}$ is particularly suited to study the representation theory of GL_n. Indeed, evaluation on the standard representation $\Bbbk^n$ of GL_n induces a map:

$$\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(F, G) \to \mathrm{Ext}^*_{GL_n}(F(\Bbbk^n), G(\Bbbk^n)) ,$$

which is an isomorphism as soon as $n \geq \max\{\deg F, \deg G\}$ [11, Cor 3.13]. Thus, one can use the powerful computational tools available in $\mathcal{P}_{\Bbbk}$ to compute the "stable" (that is, when n is big enough) extension groups between rational GL_n-modules.

A successful application of strict polynomial functors is the computation of extension groups between representations involving "Frobenius twists". If V is a rational representation of GL_n, we denote by $V^{(r)}$ the rational representation of GL_n obtained by twisting along the r-th power of the Frobenius morphism. The functor $I^{(r)} : V \mapsto V^{(r)}$ is actually a strict polynomial functor. It has a crucial role in many problems, for example in cohomological finite generation problems [11, 13, 17]. Twisted representations are also related [5] to the cohomology of the finite groups $GL_n(\mathbb{F}_q)$. For these reasons, extension groups between twisted functors (that is, extension groups of the form $\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(F \circ I^{(r)}, G \circ I^{(r)})$) have received much attention, and many successful computations have been performed [3, 4, 8, 11]. In particular, Chałupnik has proved in [3, Thm 4.3] that Ext-groups of the form $\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(\Gamma^{\mu} \circ I^{(r)}, F \circ I^{(r)})$ (where Γ^{μ} denotes a tensor product of divided powers) can be easily computed via an isomorphism:

$$\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(\Gamma^{\mu} \circ I^{(r)}, F \circ I^{(r)}) \simeq \mathrm{Hom}_{\mathcal{P}_{\Bbbk}}(\Gamma^{\mu} \circ (E_r \otimes I), F) \qquad (*)$$

where E_r is the graded vector space $\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(I^{(r)}, I^{(r)})$, and $E_r \otimes I$ denotes the graded functor $V \mapsto E_r \otimes V$.

In this article, we give a new approach of Ext-computations between twisted functors. This approach does not depend on the earlier computations of [3, 4, 8, 11]. As main tool, we use the explicit injective coresolutions of twisted symmetric powers built by Troesch in [18]. These coresolutions generalize to all prime characteristic what was previously known in characteristic $p = 2$ only [9, 11]. At first sight these coresolutions are quite big and complicated (especially in odd characteristic), and one could fear that they are useless for concrete computations. But this is not the case: we show that only a very little part of the information contained in these coresolutions is needed for computations. In particular, we do not need the information borne by the differential (see Lemma 4.4)! We exploit the latter fact to get the first main result of the article, namely:

– we get a new and simpler proof of Chałupnik's isomorphism $(*)$, and we derive from this isomorphism new short proofs of many Ext-computations.

Then we try to go further in the study of extension groups between twisted functors. With the help of Troesch complexes, we obtain new results in two independent directions.

– First, we apply isomorphism $(*)$ to compute rational cohomology algebras of GL_n. To do this, we need to improve Friedlander and Suslin's bound on n so that $\mathrm{Ext}^*_{\mathscr{P}_\Bbbk}(\Gamma^\mu \circ I^{(r)}, F \circ I^{(r)})$ computes GL_n extensions. As an application, we give explicit generators and relations for the cohomology algebra $H^*(GL_n, A)$, where A is an algebra of polynomials over a direct sum of copies of the twisted standard representation $(\Bbbk^n)^{(r)}$ and its dual (for n big enough) in the spirit of classical invariant theory. We give a conjecture for smaller n.
– We introduce in Section 7 the "twisting spectral sequence" which generalizes isomorphism $(*)$. We show that this spectral sequence contains interesting information about extensions between twisted functors. As main new result, we prove in Section 8 that in many cases this spectral sequence collapses at the second page, and we conjecture that this is a general fact. A positive answer to this conjecture would improve significantly our understanding of the homological effects of Frobenius twists.

Let us review more specifically the content of the paper. Sections 2 and 3 are mainly expository. They collect well-know facts about $\mathscr{P}_\Bbbk$ and describe the properties of Troesch coresolutions which we need for our computations (in Propositions 3.4 and 3.7). For our computations, we do not need an explicit description of the differentials of Troesch coresolutions. However, for the reader's convenience, we have recalled their construction in an appendix.

In Section 4, we present our elementary proof of Chałupnik's isomorphism $(*)$. As corollaries, we retrieve many computations from [3, 8, 11]. Beside brevity, our proofs have the advantage to avoid the use of many technical tools (e.g. functors with several variables, generalized Koszul complexes, trigraded Hopf algebra structures on hypercohomology spectral sequences, symmetrizations of functors) which seemed essential in [8, 11], and also in [3] since this latter article elaborates on the results of [8].

In Section 5, we enrich the computation of $\mathrm{Ext}^*_{\mathscr{P}_\Bbbk}(\Gamma^\mu \circ I^{(r)}, F \circ I^{(r)})$ by describing cup products as well as the twisting map:

$$\mathrm{Fr}_1 : \mathrm{Ext}^*_{\mathscr{P}_\Bbbk}(\Gamma^\mu \circ I^{(r)}, F \circ I^{(r)}) \to \mathrm{Ext}^*_{\mathscr{P}_\Bbbk}(\Gamma^\mu \circ I^{(r+1)}, F \circ I^{(r+1)})$$

induced by precomposition by $I^{(1)}$. Although the result might not surprise experts, it is neither stated, nor proved in the literature. We use it to generalize some Hopf algebra computations of [8].

In Section 6, we try to apply our Ext-computations in $\mathscr{P}_\Bbbk$ to compute some rational cohomology algebras for GL_n. When doing so, we encounter the problem that Friedlander and Suslin's bound on n such that $\mathrm{Ext}^*_{\mathscr{P}_\Bbbk}(F \circ I^{(r)}, G \circ I^{(r)})$ computes GL_n extensions is not sufficient. Fortunately, with the help of Troesch complexes, we prove that this bound can be substantially improved for extensions of the form $\mathrm{Ext}^*_{\mathscr{P}_\Bbbk}(\Gamma^\mu \circ I^{(r)}, F \circ I^{(r)})$. Combining this with the previous computations of Section 5 we prove a cohomological analogue of the "Fundamental Theorems" from classical invariant theory. Namely, we describe the cohomology algebra $H^*(GL_n, S^*((\Bbbk^{n\,(r)})^{\oplus k} \oplus (\Bbbk^{n\,(r)})^{\vee\,\oplus\ell})$. Actually our result is valid for $n \geq p^r \min\{k,\ell\}$. For n smaller, we state a conjecture.

We introduce the "twisting spectral sequence" $E^{s,t}(F,G,r)$ in Section 7. The second page of this spectral sequence is given by extension groups between F and G precomposed by the functor $V \mapsto E_r \otimes V$, and it converges to the extension groups $\mathrm{Ext}^*_{\mathscr{P}_\Bbbk}(F \circ I^{(r)}, G \circ I^{(r)})$. Although the construction of the twisting spectral sequence is a formal consequence of Chałupnik's isomorphism, it is an interesting tool to study extensions between twisted functors. Indeed, the effect of cup products and the effect of the twisting map Fr_1 may be easily read on the second page, so the twisting spectral sequence is a convenient way to study them. For example, we can read the "twist stability" phenomenon [8, Cor 4.10] on the second page, and the injectivity of the twisting map Fr_1 is implied by the collapsing at the second page. Also, many classical computations amount to the collapsing (for lacunary reasons) of the twisting spectral sequence at the second page.

In fact, we observe that the twisting spectral sequence collapses at the second page in all the computations known so far, even when there is no lacunary reason for it, and we conjecture this is a general fact. As a main result, we make a step towards this conjecture, by proving in Section 8 that $E(F,G,r)$ collapses at the second page for all $r \geq 0$ and many pairs (F,G), including all pairs (F,G) studied in [3, 4, 8, 11]. We also propose to the reader a combinatorial problem whose positive solution would prove the collapsing for any F, G.

2. Background and notations

In the article, we assume from the reader only a basic knowledge of the category $\mathscr{P}_\Bbbk$, corresponding to Section 2 of the seminal article [11]. We do not assume that any Ext-computation is known (we redo all the computations from scratch). In this background section, we introduce notations (most of them are standard), and we write down a few useful facts which are either implicit in, or easy consequences of [11, Section 2].

2.1. Notations

Throughout the article, $\Bbbk$ is a field of prime characteristic p. If V is a $\Bbbk$-vector space, we let $V^\vee := \mathrm{Hom}_\Bbbk(V, \Bbbk)$. Many notations are as in [11]. In particular $F^\sharp(V) := F(V^\vee)^\vee$ denotes the dual of a functor F (as in [11, Prop. 2.6]) and $F^{(r)}$ denotes the composite $F \circ I^{(r)}$. For the sake of simplicity, we drop the index "$\mathscr{P}_\Bbbk$" on Hom and Ext-groups when no confusion is possible (i.e., $\mathrm{Ext}^*(F,G)$ means $\mathrm{Ext}^*_{\mathscr{P}_\Bbbk}(F,G)$).

We denote tuples of nonnegative integers by Greek letters λ, μ, ν. Let $\mu = (\mu_1, \dots, \mu_n)$ be a tuple. The weight of μ is the integer $\sum \mu_i$. If m is an integer, we denote by $m\mu$ the tuple $(m\mu_1, \dots, m\mu_n)$. We say that a positive integer d divides μ if for all $1 \leq j \leq n$, d divides μ_j. If X is one of the symbols S, Λ, Γ, we denote by X^μ the tensor product $X^{\mu_1} \otimes \cdots \otimes X^{\mu_n}$ (since $X^0 = \Bbbk$, this tensor product has a meaning even if some μ_i are zero).

We denote by I the identity functor ($I = \Lambda^1 = \Gamma^1 = S^1$), and if W is a finite dimensional vector space we denote by $F(W \otimes I)$ the precomposition of F by the functor $V \mapsto W \otimes V$.

Finally, we denote by $ш_r$ the finite dimensional graded vector space which equals $\Bbbk$ in degrees i for $0 \leq i < p^r$ and which is zero in the other degrees (we use the Cyrillic letter "sha" by analogy with a Dirac comb). We also denote by E_r the even degree version of $ш_r$,