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KAROUBI'S RELATIVE CHERN CHARACTER AND BEILINSON'S REGULATOR

BY GEORG TAMME

ABSTRACT. – We construct a variant of Karoubi's relative Chern character for smooth varieties over **C** and prove a comparison result with Beilinson's regulator with values in Deligne-Beilinson cohomology. As a corollary we obtain a new proof of Burgos' Theorem that for number fields Borel's regulator is twice Beilinson's regulator.

RÉSUMÉ. – Nous construisons une variante du caractère de Chern relatif de Karoubi pour les variétés lisses sur **C** et prouvons un résultat de comparaison avec le régulateur de Beilinson à valeurs dans la cohomologie de Deligne-Beilinson. En corollaire, nous obtenons une nouvelle preuve du théorème de Burgos que, pour un corps de nombres, le régulateur de Beilinson est deux fois le régulateur de Borel.

Introduction

In a series of papers [25, 26, 27, 29] Karoubi introduced relative K-theory for Banach algebras A (the homotopy fibre of the map from algebraic to topological K-theory) and constructed the relative Chern character

$$\mathrm{Ch}_i^{\mathrm{rel}}\colon K_i^{\mathrm{rel}}(A) \to HC_{i-1}^{\mathrm{cont}}(A)$$

mapping relative K-theory to continuous cyclic homology. He also mentioned a geometric version of his relative Chern character and the possible connection with regulators.

The objective of this paper is to make these relations precise in the case of smooth affine varieties X over **C**. In this situation the cyclic homology decomposes into a product of cohomology groups of the truncated de Rham complex and the relative Chern character becomes a morphism

$$\mathrm{Ch}_{n,i}^{\mathrm{rel}}\colon K_i^{\mathrm{rel}}(X) \to \mathbb{H}^{2n-i-1}(X, \Omega_X^{<n}).$$

We may formulate our main result as follows.

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Theorem. – *Karoubi's relative Chern character factors naturally through a morphism*

$$K_i^{\mathrm{rel}}(X) \to H^{2n-i-1}(X, \mathbf{C})/\mathrm{Fil}^n H^{2n-i-1}(X, \mathbf{C})$$

which we denote by the same symbol. The diagram

$$\begin{tikzcd} K_i^{\mathrm{rel}}(X) \arrow[rr] \arrow[d, "\mathrm{Ch}_{n,i}^{\mathrm{rel}}"] & & K_i(X) \arrow[d, "\mathrm{Ch}_{n,i}^{\mathscr{D}}"] \\ H^{2n-i-1}(X, \mathbf{C})/\mathrm{Fil}^n H^{2n-i-1}(X, \mathbf{C}) \arrow[rr] & & H_{\mathscr{D}}^{2n-i}(X, \mathbf{Q}(n)), \end{tikzcd}$$

where $\mathrm{Ch}_{n,i}^{\mathscr{D}}$ *is Beilinson's Chern character with values in Deligne-Beilinson cohomology, commutes.*

As an application we give a new proof for the comparison of Borel's and Beilinson's regulators—the case $X = \mathrm{Spec}(\mathbf{C})$:

Corollary (Burgos' Theorem [6]). – *Borel's regulator*

$$K_{2n-1}(\mathbf{C}) \to \mathbf{R}(n-1)$$

is twice Beilinson's regulator.

This result plays an important role in the study of special values of L-functions: Borel [4] established a precise relation between his regulator and special values of zeta functions of number fields. In [1] Beilinson formulated far reaching conjectures describing special values of L-functions of motives up to non-zero rational factors in terms of his regulator. He also proved that for a number field his regulator coincides with Borel's up to a non-zero rational factor (see also Rapoport's report [32]). This enabled him to view Borel's computations as a confirmation of his conjectures in the case of a number field.

However, in order to exploit Borel's result further and remove the $\mathbf{Q}^\times$-ambiguity it is important to have a precise comparison result for the regulators as provided by Burgos' Theorem.

In [12] Dupont, Hain, and Zucker proposed a strategy for the comparison of both regulators based on the comparison of Cheeger-Simons' and Beilinson's Chern character classes. While there remained some difficulties in carrying this out, they were led to the conjecture that the precise factor would be 2. This was then proven by Burgos using Beilinson's original argument.

So far Karoubi's relative Chern character has not much been studied in Arithmetics. One of its possible advantages is that it is defined in complete analogy in the classical, p-adic, and even non commutative situation and thus gives a unifying frame for the study of regulators in these different contexts. In the p-adic setting analogues of the results presented here have been obtained in [37, 38].

Karoubi's principal idea is to describe relative K-theory in terms of bundles with discrete structure group on certain simplicial sets together with a trivialization of the associated topological bundle. The relative Chern character is induced by secondary classes for these bundles constructed by means of Chern-Weil theory.

The first problem one encounters when trying to compare these classes with Beilinson's is that they live in the cohomology of the truncated de Rham complex which does not map naturally to Deligne-Beilinson cohomology. It is therefore necessary to construct refinements of these classes which are then to be compared with the corresponding classes in Deligne-Beilinson cohomology.

Our approach to this is to generalize Karoubi's formalism to simplicial manifolds and systematically use what we call *topological morphisms* and *bundles*. The Chern-Weil theoretic construction of secondary classes in this setup is described in Section 1. In the second section we make essential use of the notion of topological morphisms in order to construct the above mentioned refinements (Proposition 2.10) and compare them with Beilinson's classes (Theorem 2.11).

Our construction of the relative Chern character on K-theory is presented in Section 3. It differs slightly from Karoubi's original one. The comparison with Beilinson's regulator then follows formally from the results of the second section.

The corollary is finally proven in Section 4. By the theorem it reduces to a comparison of Karoubi's relative Chern character for $X = \mathrm{Spec}(\mathbf{C})$ and Borel's regulator. A similar result has been obtained previously by Hamida [22]. She constructs an explicit map $K_{2n-1}(\mathbf{C}) \to K^{\mathrm{rel}}_{2n-1}(\mathbf{C})$ and composes it with the relative Chern character to obtain a map defined on the K-theory of $\mathbf{C}$ rather than the relative K-theory. This is then compared with Borel's regulator.

Related to ours is the work of Soulé [35] who showed that Beilinson's Chern character $\mathrm{Ch}^{\mathscr{D}}_{n,i}\colon K_i(X) \to H^{2n-i}_{\mathscr{D}}(X, \mathbf{Q}(n))$ factors through the *multiplicative* K*-theory* of X which may also be described in terms of bundles and connections on certain simplicial algebraic varieties. However, the relation with the relative Chern character or Borel's regulator is not treated in that paper.

The results presented here are part of my Ph.D. thesis [36] at the Universität Regensburg. It is a pleasure to thank my advisor Guido Kings for his guidance. I would like to thank the California Institute of Technology, where this paper was completed, and especially Matthias Flach for their hospitality. Finally, I would like to thank the referee for his or her careful reading of the manuscript and several valuable suggestions improving the exposition of the paper.

Notation

For a complex manifold Y the sheaves of holomorphic functions, holomorphic n-forms, and smooth n-forms are denoted by $\mathscr{O}_Y, \Omega^n_Y$, and $\mathscr{A}^n_Y$, respectively. Global sections are denoted by $\Omega^n(Y)$, etc.

The ordered set $\{0, \dots, p\}$ is denoted by $[p]$. Simplicial objects are usually marked with a bullet like $X_\bullet$. The i^{th} face and degeneracy of a simplicial object are denoted by ∂_i and s_i, respectively. The i^{th} coface of a cosimplicial object is denoted by δ^i. The geometric realization of a simplicial set $S_\bullet$ is denoted by $|S_\bullet|$.

If $f\colon A^\bullet \to B^\bullet$ is a morphism of cochain complexes, we define $\mathrm{Cone}(f)$ to be the complex given in degree n by $A^{n+1} \oplus B^n$ with $d(a,b) = (-da, db - f(a))$. The complex $A[-1]$ is given in degree n by A^{n-1} with differential $-d$.

1. Karoubi's secondary classes

1.1. De Rham cohomology of simplicial complex manifolds

Here we recall Dupont's computation of the de Rham cohomology of simplicial manifolds [10] in the setting of complex manifolds. This is fundamental for simplicial Chern-Weil theory in the following sections.

Let $X_\bullet$ be a simplicial complex manifold and denote by $\Omega^{\geq r}_{X_\bullet}$ for $r \geq 0$ the naively truncated de Rham complex of sheaves of holomorphic forms, i.e. the r^{th} step of the bête filtration. Then we have

$$\mathbb{H}^*(X_\bullet, \Omega^{\geq r}_{X_\bullet}) = H^*(\operatorname{Tot}\operatorname{Fil}^r \mathscr{A}^*(X_\bullet)),$$

where $\operatorname{Tot}\operatorname{Fil}^r \mathscr{A}^*(X_\bullet)$ is the total complex associated with the cosimplicial complex $[p] \mapsto \operatorname{Fil}^r \mathscr{A}^*(X_p) = \bigoplus_{k+l=*, k\geq r} \mathscr{A}^*(X_p)$ (cf. [9, (5.2.7)]). For the purpose of simplicial Chern-Weil theory we need another version of the simplicial de Rham complex. Let

$$\Delta^p := \left\{(x_0, \dots, x_p) \in \mathbf{R}^{p+1} \;\middle|\; x_i \geq 0, \sum\nolimits_{i=0}^{p} x_i = 1\right\} \subset \mathbf{R}^{p+1}$$

denote the standard simplex. Then $[p] \mapsto \Delta^p$ is a cosimplicial space in a natural way. A function or form on Δ^p is called smooth, if it extends to a smooth function, resp. form on a neighborhood of Δ^p in $\{\sum x_i = 1\} \subset \mathbf{R}^{p+1}$. We recall from [10]:

Definition 1.1. – A smooth simplicial n-form on a simplicial complex manifold $X_\bullet$ is a family $\omega = (\omega_p)_{p\geq 0}$, where ω_p is a smooth n-form on $\Delta^p \times X_p$, and the compatibility condition

$$(\delta^i \times 1)^* \omega_p = (1 \times \partial_i)^* \omega_{p-1} \quad \text{on} \quad \Delta^{p-1} \times X_p$$

$i = 0, \dots, p$, $p \geq 0$, is satisfied. The space of smooth simplicial n-forms on $X_\bullet$ is denoted by $A^n(X_\bullet)$.

The exterior derivative d and the usual wedge product applied component-wise make $A^*(X_\bullet)$ into a commutative differential graded $\mathbf{C}$-algebra.

Moreover, $A^*(X_\bullet)$ is naturally the total complex associated with the triple complex $(A^{k,l,m}(X_\bullet), d_\Delta, \partial_X, \bar\partial_X)$ where $A^{k,l,m}(X_\bullet)$ consists of the forms ω of type (k,l,m), that is, each ω_p is locally of the form $\sum_{I,J,K} f_{I,J,K} dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge d\zeta_{j_1} \wedge \cdots \wedge d\zeta_{j_l} \wedge d\bar\zeta_{k_1} \wedge \cdots \wedge d\bar\zeta_{k_m}$, where $x_0, \dots, x_p$ are the barycentric coordinates on Δ^p and the ζ_j are *holomorphic* coordinates on X_p. $d_\Delta, \partial_X, \bar\partial_X$ denote the exterior derivative in Δ-and the Dolbeault-derivations in X-direction, respectively. Write $\operatorname{Fil}^r A^*(X_\bullet) = \bigoplus_{k+l+m=*, l\geq r} A^{k,l,m}(X_\bullet)$.

On the other hand we have the triple complex $(\mathscr{A}^{k,l,m}(X_\bullet), \delta, \partial_X, \bar\partial_X)$, where $\mathscr{A}^{k,l,m}(X_\bullet) = \mathscr{A}^{l,m}(X_k)$ and $\delta = \sum_{i=0}^{k} (-1)^i \partial_i^* : \mathscr{A}^{k,l,m}(X_\bullet) \to \mathscr{A}^{k+1,l,m}(X_\bullet)$.

Theorem 1.2 (Dupont). – *Let $X_\bullet$ be a simplicial complex manifold. For each $l, m \geq 0$ the two complexes $(A^{*,l,m}(X_\bullet), d_\Delta)$ and $(\mathscr{A}^{*,l,m}(X_\bullet), \delta)$ are naturally chain homotopy equivalent. The equivalence is given by integration over the standard simplex:*

$$I\colon A^{k,l,m}(X_\bullet) \to \mathscr{A}^{k,l,m}(X_\bullet), \quad \omega = (\omega_p)_{p\geq 0} \mapsto \int_{\Delta^k} \omega_k.$$

In particular, we get natural isomorphisms

$$\mathbb{H}^*(X_\bullet, \Omega^{\geq r}_{X_\bullet}) \cong H^*(\operatorname{Tot}\operatorname{Fil}^r \mathscr{A}^*(X_\bullet)) \cong H^*(\operatorname{Fil}^r A^*(X_\bullet)).$$