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ON SCHRÖDINGER MAPS FROM T^1 TO S^2

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ABSTRACT. – We prove an estimate for the difference of two solutions of the Schrödinger map equation for maps from T^1 to S^2. This estimate yields some continuity properties of the flow map for the topology of $L^2(T^1, S^2)$, provided one takes its quotient by the continuous group action of T^1 given by translations. We also prove that without taking this quotient, for any $t > 0$ the flow map at time t is discontinuous as a map from $\mathscr{C}^\infty(T^1, S^2)$, equipped with the weak topology of $H^{1/2}$, to the space of distributions $(\mathscr{C}^\infty(T^1, \mathbb{R}^3))^*$. The argument relies in an essential way on the link between the Schrödinger map equation and the binormal curvature flow for curves in the euclidean space, and on a new estimate for the latter.

RÉSUMÉ. – Nous obtenons une estimation de la différence entre deux solutions de l'équation des Schrödinger maps de T^1 dans S^2. Cette estimation fournit une propriété de continuité du flot associé pour la topologie de $L^2(T^1, S^2)$, quotientée par l'action continue du groupe T^1 via les translations. Nous démontrons également que sans cette prise de quotient, quel que soit $t > 0$ l'application flot au temps t est discontinue de $\mathscr{C}^\infty(T^1, S^2)$, équipé de la topologie faible de $H^{1/2}$, vers l'espace des distributions périodiques $(\mathscr{C}^\infty(T^1, \mathbb{R}^3))^*$. L'argument repose de manière essentielle sur le lien étroit entre l'équation des Schrödinger maps et celle du flot par courbure binormale pour une courbe dans l'espace, et sur une nouvelle estimation concernant ce dernier.

1. Introduction

We are interested in the Schrödinger map equation,

$$\partial_t u = \partial_s (u \times \partial_s u)\,, \tag{1}$$

where $u : I \times T^1 \to S^2$, $0 \in I \subset \mathbb{R}$ is some open interval, $T^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$ is the flat one dimensional torus, S^2 is the standard unit sphere in $\mathbb{R}^3$, and $\times$ is the vector product in $\mathbb{R}^3$. The solutions we consider are all at least in $L^\infty(I, H^{1/2}(T^1, S^2))$, so that Equation (1) has a well-defined distributional meaning in $I \times T^1$. Such solutions are necessarily continuous with values into $H^{1/2}_{\text{weak}}(T^1, S^2)$, so that Cauchy problems are well-defined too.

Our main result provides an upper bound for the growth rate of the difference between two solutions, one of which being required to be sufficiently smooth.

THEOREM 1. – *Let* $u \in \mathscr{C}(I, H^3(T^1, S^2))$ *be a solution of the Schrödinger map Equation* (1) *on* $I = (-T, T)$ *for some* $T > 0$. *Given any other solution* $v \in L^\infty(I, H^{1/2}(T^1, S^2))$ *of* (1), *there exists a continuous function* $\sigma : I \to T^1$ *such that for every* $t \in I$,

$$\|v(t,\cdot) - u(t, \cdot + \sigma(t))\|_{L^2(T^1,\mathbb{R}^3)} \leq C\|v(0,\cdot) - u(0,\cdot)\|_{L^2(T^1,\mathbb{R}^3)},$$

where the constant $C \equiv C(\|\partial_{sss}u(0,\cdot)\|_{L^2}, T)$, *in particular* C *does not depend on* v.(1)

For initial data in $H^k(T^1, S^2)$, with $k \geq 2$, Equation (1) possesses unique global solutions in $\mathscr{C}(\mathbb{R}, H^k(T^1, S^2))$. Those solutions preserve the energy

$$E(u(t,\cdot)) := \int_{T^1} \left|\partial_s u\right|^2(t,s)\,ds.$$

For initial data in the energy space $H^1(T^1, S^2)$, global solutions in $L^\infty(\mathbb{R}, H^1(T^1, S^2))$ are also known to exist. The latter are continuous with values in $H^1_{\text{weak}}(T^1, S^2)$, but are not known to be unique nor to preserve the energy.

REMARK 1. – *Let* $D(\mathscr{F}) \subseteq H^{1/2}(T^1, S^2))$ *be the domain of a hypothetical flow map* $\mathscr{F}$ *for* (1) *with values into* $\mathscr{C}(I, H^{1/2}_{\text{weak}}(T^1, S^2))$, *and let* $\mathscr{G}$ *be one of the (possibly unique) existing flow maps from* $H^1(T^1, S^2)$ *with values into* $\mathscr{C}(I, H^1_{\text{weak}}(T^1, S^2))$. *For* $s \geq 0$, *let* H^s/S^1 *and* H^s_{weak}/S^1 *denote the topological spaces obtained as the quotients of* $H^s(T^1, \mathbb{R}^3)$ *and* $H^s_{\text{weak}}(T^1, \mathbb{R}^3)$ *respectively by the continuous action group of* S^1 *through translations*(2). *Let finally* $\mathscr{F}_{/S^1}$ *and* $\mathscr{G}_{/S^1}$ *denote the left composition of* $\mathscr{F}$ *and* $\mathscr{G}$ *respectively by the above quotient map*(3). *As a consequence of Theorem 1, we see that:*

At any point in $H^3(T^1, S^2)$, $\mathscr{F}_{/S^1}$ *is continuous as a map*

$$\mathscr{F}_{/S^1} \;:\; \Big(D(\mathscr{F})\,,\, \mathrm{top}\big(L^2\big) \Big) \longrightarrow \Big(\mathscr{C}\big(H^{1/2}_{\text{weak}}\big)\,,\, \mathrm{top}\big(\mathscr{C}(L^2/T^1)\,\big)\Big),$$

and $\mathscr{G}_{/S^1}$ *is continuous as a map*

$$\mathscr{G}_{/S^1} \;:\; \Big(H^1\,,\, \mathrm{top}\big(H^1_{\text{weak}}\big) \Big) \longrightarrow \Big(\mathscr{C}\big(H^1_{\text{weak}}\big)\,,\, \mathrm{top}\big(\mathscr{C}(H^1_{\text{weak}}/T^1)\,\big)\Big).$$

We supplement Theorem 1 and Remark 1 by the following ill-posedness type result, which shows in particular that the function σ in the statement of Theorem 1 could not be removed.

THEOREM 2. – *For any given* $t \neq 0$, *the flow map for* (1) *at time* t *is not continuous as a map from* $\mathscr{C}^\infty(T^1, S^2)$, *equipped with the weak topology of* $H^{1/2}$, *to the space of distributions* $(\mathscr{C}^\infty(T^1, \mathbb{R}^3))^*$. *Indeed, for any* $\sigma_0 \in \mathbb{R}$ *there exist a sequence of smooth initial data* $(u_{m,\sigma_0}(0,\cdot))_{m\in\mathbb{N}} \in \mathscr{C}^\infty(T^1, S^2)$ *such that*

$$u_{m,\sigma_0}(0,\cdot) \rightharpoonup u^*(0,\cdot) \qquad\qquad \textit{in } H^{\frac{1}{2}}_{\text{weak}}(T^1, \mathbb{R}^3),$$

(1) See Section 7 for a more explicit version of C.
(2) These are of course no longer vector spaces.
(3) We mean pointwise in I. In other words $\mathscr{F}_{/S^1}$ and $\mathscr{G}_{/S^1}$ describe the profiles of the solutions, independently of their locations.

where $u^*(0,s) := (\cos(s), \sin(s), 0)$ *is a stationary solution of* (1), *and for any* $t \in \mathbb{R}$

$$u_{m,\sigma_0}(t,\cdot) \rightharpoonup u^*(t,\cdot + \sigma_0 t) \qquad \text{in } H^{\frac{1}{2}}_{\text{weak}}(T^1, \mathbb{R}^3).$$

1.1. Three companions : Binormal flow, Schrödinger map, Cubic NLS

The binormal curvature flow equation for $\gamma : I \times \mathbb{R} \to \mathbb{R}^3$, where $0 \in I \subset \mathbb{R}$ is some open interval, is given by

$$\partial_t \gamma = \partial_s \gamma \times \partial_{ss} \gamma, \tag{2}$$

where s is moreover required to be an arc-length parameter for the curves $\gamma(t,\cdot)$, for every $t \in I$. At least for smooth solutions, the arc-length parametrization condition $|\partial_s \gamma(t,s)|^2 = 1$ is compatible with Equation (2), since

$$\partial_t\big(|\partial_s\gamma|^2\big) = 2\partial_s\gamma \cdot \partial_{st}\gamma = 2\partial_s\gamma \cdot \big(\partial_s\gamma \times \partial_{sss}\gamma\big) = 0$$

whenever (2) is satisfied. Denoting by $(\tau, n, b) \equiv (\tau(t,s), n(t,s), b(t,s))$ the Frenet-Serret frame along the curve $\gamma(t,\cdot)$ at the point $\gamma(t,s)$, so that $\partial_s\gamma = \tau$ and $\partial_{ss}\gamma = \kappa n$ where κ is the curvature, Equation (2) may be rephrased as

$$\partial_t \gamma = \kappa b,$$

from which its name arises. In the sequel, we will only consider solutions of (2) which belong to $L^\infty(I, H^{3/2}_{loc}(\mathbb{R}, \mathbb{R}^3))$, so that (2) has a well-defined distributional meaning in $I \times \mathbb{R}$.

If $\gamma \in L^\infty(I, H^{3/2}_{loc}(\mathbb{R}, \mathbb{R}^3))$ is a solution to the binormal curvature flow Equation (2), then the map $u := \partial_s \gamma \in L^\infty(I, H^{1/2}_{loc}(\mathbb{R}, S^2))$, parametrizing the evolution in time of the unit tangent vectors to the curves $\gamma(t,\cdot)$, satisfies

$$\partial_t u = \partial_t \partial_s \gamma = \partial_s \partial_t \gamma = \partial_s\big(u \times \partial_s u\big)$$

in the sense of distributions on $I \times \mathbb{R}$. In other words, u is a solution of the Schrödinger map Equation (1) for maps from $\mathbb{R}$ to S^2, and the binormal curvature flow equation is therefore a primitive equation of the Schrödinger map equation.

Conversely, let $u \in L^\infty(I, H^{1/2}_{loc}(\mathbb{R}, S^2))$ be a solution to the Schrödinger map Equation (1) and define the function $\Gamma_u \in L^\infty(I, H^{3/2}_{loc}(\mathbb{R}, \mathbb{R}^3))$ by

$$\Gamma_u(t,s) := \int_0^s u(t,z)\, dz. \tag{3}$$

In the sense of distributions on $I \times \mathbb{R}$, we have

$$\partial_s\Big(\partial_t \Gamma_u - \partial_s \Gamma_u \times \partial_{ss} \Gamma_u\Big) = 0. \tag{4}$$

By construction, the primitive curves $\Gamma_u(t,\cdot)$ all have their base point $\Gamma_u(t,0)$ fixed at the origin. If they were smooth, Equation (4) would directly imply the existence of a function c_u depending on time only and such that

$$\gamma_u(t,s) := \Gamma_u(t,s) + c(t)$$

is a solution to the binormal curvature flow Equation (2). The function c_u indeed represents the evolution in time of the actual base point of the curves. In Section 2 we will turn this argument into a statement for quasiperiodic solutions in $H^{3/2}$.

Our proofs of Theorem 1 and Theorem 2 spend most of their time in the binormal curvature flow world rather than in the Schrödinger map one.

Besides the bidirectional relation between the binormal curvature flow equation and the Schrödinger map equation presented just above, Hasimoto [9] exhibited in 1972 an intimate relation between the binormal curvature flow Equation (2) and the cubic focusing nonlinear Schrödinger equation. Let $\gamma : I \times \mathbb{R} \to \mathbb{R}^3$ be a smooth and biregular solution of the binormal curvature flow Equation (2), and denote by κ and T respectively the curvature and torsion functions of γ. Then, the function Ψ defined on $I \times \mathbb{R}$ by the Hasimoto transform

$$\psi(t,s) := \kappa(t,s) \exp\left(i\int_0^s T(t,z)\,dz\right)$$

is a solution to

$$\partial_t\psi + \partial_{ss}\psi + \frac{1}{2}(|\psi|^2 - A(t))\psi = 0$$

where

$$A(t) := \left(2\frac{\partial_{ss}\kappa - \kappa T^2}{\kappa} + \kappa^2\right)(t,0).$$

If γ is 2π-periodic in s, that is if $\gamma : I \times T^1 \to \mathbb{R}^3$, then Ψ is only quasiperiodic unless $\int_0^{2\pi} T(t,z)\,dz \in 2\pi\mathbb{Z}$. Nevertheless, it is possible to recover a 2π-periodic function Ψ by means of a Galilean transform. One can also get rid of the $A(t)$ factor by means of a phase shift. More precisely, the function

$$\Psi(t,s) := \psi(t, s - \frac{b}{2}t) \exp\left(i(bs - b^2t - \int_0^t \frac{1}{2}A(z)\,dz)\right),$$

where

$$b := 1 - \frac{1}{2\pi}\int_0^{2\pi} T(0,z)\,dz = 1 - \frac{1}{2\pi}\int_0^{2\pi} T(t,z)\,dz$$

is well-defined on $I \times T^1$ and is a solution to the cubic focusing nonlinear Schrödinger equation

$$\partial_t\Psi + \partial_{ss}\Psi + \frac{1}{2}|\Psi|^2\Psi = 0 \tag{5}$$

on $I \times T^1$.

Equation (5) is integrable and known to be solvable by the inverse scattering method since the works of Zakharov and Shabat [21] in 1971 for the vanishing case and Ma and Ablowitz [14] in 1981 for the periodic case. Therefore the binormal curvature flow Equation (2) and the Schrödinger map Equation (1) are also integrable, in the weak sense that they can be mapped to an integrable equation. Notice however that the inverse of the Hasimoto transform, whenever it is well defined[(4)], involves reconstructing a curve from its curvature and torsion functions, an operation which is both highly nonlinear and nonlocal. At least for that reason, translating an estimate available for the cubic focusing NLS to the binormal curvature flow equation or to the Schrödinger map equation, or even vice versa, does not look like a straightforward task.

[(4)] Vanishing of Ψ yields underdetermination.