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BASE CHANGE FOR BERNSTEIN CENTERS OF DEPTH ZERO PRINCIPAL SERIES BLOCKS

BY THOMAS J. HAINES

ABSTRACT. – Let G be an unramified group over a p-adic field. This article introduces a base change homomorphism for Bernstein centers of depth-zero principal series blocks for G and proves the corresponding base change fundamental lemma. This result is used in the approach to Shimura varieties with $\Gamma_1(p)$-level structure initiated by M. Rapoport and the author in [14].

RÉSUMÉ. – Soit G un groupe non-ramifié sur un corps p-adique. On définit un homomorphisme de changement de base pour les centres de Bernstein des blocs des séries principales de niveau zéro, et on démontre le lemme fondamental correspondant. Ce résultat est utilisé dans le calcul du facteur local en p des variétés de Shimura à structure de niveau $\Gamma_1(p)$ dans l'article avec M. Rapoport [14] publié en tandem avec cet article dans ce même journal.

1. Introduction

Let F denote a p-adic field, and let $F_r \supset F$ denote the unique degree r unramified extension of F contained in some algebraic closure $\bar{F}$ of F. Let θ denote a generator of $\mathrm{Gal}(F_r/F)$. Let G denote an unramified connected reductive group over F. The automorphism θ determines an automorphism of $G(F_r)$, which we also denote with the symbol θ.

Using θ, we have the notion of stable twisted orbital integral $\mathrm{SO}_{\delta\theta}(\phi)$ for any locally constant compactly-supported function ϕ on $G(F_r)$ and any element $\delta \in G(F_r)$ with semisimple norm. See [17] for the definition of the norm map $\mathcal{N}$ from stable θ-conjugacy classes in $G(F_r)$ to stable conjugacy classes in $G(F)$. For a precise definition of $\mathrm{SO}_{\delta\theta}$, see e.g. [18], or [11, 5.1.2].

This article is concerned with the matching of the (twisted) orbital integrals of certain functions on the groups $G(F_r)$ and $G(F)$, respectively. If $\phi \in \mathcal{H}(G(F_r))$ and $f \in \mathcal{H}(G(F))$ are functions in the corresponding Hecke algebras of locally constant compactly-supported functions, then we say ϕ, f are *associated* (or *have matching orbital integrals*), if the following

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result holds for the stable (twisted) orbital integrals: for every semisimple element $\gamma \in G(F)$, we have

$$\mathrm{SO}_\gamma(f) = \sum_\delta \Delta(\gamma, \delta)\mathrm{SO}_{\delta\theta}(\phi)$$

where the sum is over stable θ-conjugacy classes $\delta \in G(F_r)$ with semisimple norm, and where $\Delta(\gamma, \delta) = 1$ if $\mathcal{N}\delta = \gamma$ and $\Delta(\gamma, \delta) = 0$ otherwise. See e.g. [18], [19], [7], or [11] for further discussion.

Of primary importance is the case of spherical Hecke algebras. Suppose $K_r \subset G(F_r)$ and $K \subset G(F)$ are hyperspecial maximal compact subgroups associated to a hyperspecial vertex in the Bruhat-Tits building $\mathcal{B}(G(F))$ for $G(F)$, and suppose $\phi \in \mathcal{H}(G(F_r))$ belongs to the corresponding spherical Hecke algebra $\mathcal{H}_{K_r}(G(F_r))$. The Satake isomorphism gives rise to a natural algebra homomorphism

$$b_r : \mathcal{H}_K(G(F_r)) \to \mathcal{H}_K(G(F)),$$

cf. [7]. The base change fundamental lemma for spherical functions asserts that ϕ and $b_r(\phi)$ are associated. This was proved by Clozel [7] and Labesse [24]. Even the earlier special cases of GL_2 [27] and GL_n [1] gave rise to important global and local applications, such as the existence of base-change lifts of certain automorphic representations for GL_n. The theorem of Clozel and Labesse played an important role in Kottwitz' work [20, 22, 21] on Shimura varieties with good reduction at p.

In [11] an analogous base change fundamental lemma is proved for centers of parahoric Hecke algebras. It plays a role in the study of Shimura varieties with parahoric level structure at p, see [10]. A very special case of [11] relates to the center of the Iwahori-Hecke algebra for $G(F)$. By Bernstein's theory [2], this can be viewed as the ring of regular functions on the variety of supercuspidal supports of the Iwahori block (the subcategory $\mathcal{R}_I(G)$ of the category $\mathcal{R}(G)$ of smooth representations of $G(F)$ whose objects are generated by their invariants under an Iwahori subgroup).

The purpose of this article is to generalize this result to certain other principal series blocks in the Bernstein decomposition, namely the *depth-zero* principal series blocks.

To state the theorem, we need some more notation. Denote the ring of integers of F by $\mathcal{O}$ and the residue field by k. Let A denote a maximal F-split torus, and set $T := \mathrm{Cent}_G(A)$, a maximal torus of G defined and unramified over F. Now choose an Iwahori subgroup $I \subset G(F)$ which is in good position relative to T, that is, the alcove $\mathbf{a}$ in $\mathcal{B}(G(F))$ corresponding to I is contained in the apartment $\mathcal{A}_T$ corresponding to A. Let I^+ denote the pro-unipotent radical of I.

Let $T(F)_1$ denote the maximal compact open subgroup of $T(F)$, and let $T(F)_1^+ = T(F)_1 \cap I^+$ denote its pro-unipotent radical. Throughout this article, χ will denote a depth-zero character on $T(F)_1$. This means that χ factors through a character $T(F)_1/T(F)_1^+ \to \mathbb{C}^\times$, which we also denote by χ. Via the canonical isomorphism

$$T(F)_1/T(F)_1^+ \xrightarrow{\sim} I/I^+,$$

we see that χ induces a smooth character $\rho := \rho_\chi$ on I which is trivial on I^+. Then we may consider the Hecke algebra $\mathcal{H}(G, \rho) := \mathcal{H}(G, I, \chi)$, which is defined as

$$\mathcal{H}(G, \rho) = \{f \in \mathcal{H}(G) \mid f(i_1 g i_2) = \rho^{-1}(i_1) f(g) \rho^{-1}(i_2) \ \ \forall i_1, i_2 \in I,\ \forall g \in G\}.$$

(Cf. [30].) Convolution is defined using the Haar measure which gives I volume 1. Write $\mathcal{Z}(G,\rho)$ for the center of $\mathcal{H}(G,\rho)$.

Let $\mathfrak{s} = \mathfrak{s}_\chi$ be the inertial equivalence class of the cuspidal pair $(T(F), \widetilde{\chi})_G$, for any extension of χ to a character $\widetilde{\chi} : T(F) \to \mathbb{C}^\times$. Then $\mathfrak{s}$ depends only on the relative Weyl group orbit of χ. We are concerned with the principal series Bernstein block

$$\mathcal{R}_{\mathfrak{s}}(G) = \mathcal{R}_\chi(G),$$

the category of smooth representations of $G(F)$ whose irreducible objects are constituents of some normalized principal series $i_B^G(\xi)$, where $\xi : T(F) \to \mathbb{C}^\times$ has $\xi|_{T(F)_1} = \chi$. It turns out (cf. Proposition 3.3.1) that (I, ρ_χ) is a Bushnell-Kutzko type for $\mathcal{R}_\chi(G)$, so that the Bernstein center of $\mathcal{R}_\chi(G)$ can be identified with the ring $\mathcal{Z}(G,\rho)$.

Let $N_r : T(F_r)_1 \to T(F)_1$ denote the norm homomorphism given by $t \mapsto t\,\theta(t)\cdots\theta^{r-1}(t)$. The character $\chi_r := \chi \circ N_r$ is a depth-zero character on $T(F_r)_1$, and gives rise to the Bernstein block $\mathcal{R}_{\chi_r}(G_r)$ for the p-adic group G_r, the character ρ_r on I_r, and the Hecke algebra $\mathcal{H}(G_r,\rho_r)$ with center $\mathcal{Z}(G_r,\rho_r)$. Here $G_r := G(F_r)$ and $I_r \subset G_r$ is the Iwahori subgroup corresponding to I.

In Definition 4.1.1 we define a *base change homomorphism*

$$b_r : \mathcal{Z}(G_r,\rho_r) \to \mathcal{Z}(G,\rho). \tag{1.0.1}$$

This is analogous to the base change homomorphisms defined for spherical Hecke algebras or centers of parahoric Hecke algebra (cf. [11]). Our main theorem is the following.

Theorem 1.1. – *If $\phi \in \mathcal{Z}(G_r,\rho_r)$, the functions ϕ and $b_r(\phi)$ are associated.*

Now write $\mathcal{H}(G,I^+)$ for $\mathcal{H}_{I^+}(G(F))$ and $\mathcal{Z}(G,I^+)$ for its center. In Section 10, we use (1.0.1) to define a natural algebra homomorphism

$$b_r : \mathcal{Z}(G_r,I_r^+) \to \mathcal{Z}(G,I^+). \tag{1.0.2}$$

Moreover, we show how Theorem 1.1 immediately implies the following result.

Corollary 1.2. – *If $\phi \in \mathcal{Z}(G_r,I_r^+)$, the functions ϕ and $b_r(\phi)$ are associated.*

In [14], these results are used in the special case of $G = \mathrm{GL}_d$ to study Shimura varieties in the Drinfeld case with $\Gamma_1(p)$-level structure at p. In future works they will be applied to other Shimura varieties with $\Gamma_1(p)$-level structure. Here, by "$\Gamma_1(p)$-level" we mean that the compact open subgroup K_p coming from the Shimura data $(\mathbf{G}, X, K_pK^p)$ is the pro-unipotent radical of an Iwahori subgroup of $\mathbf{G}(\mathbb{Q}_p)$. Theorem 1.1 and Corollary 1.2 will play a role in the pseudostabilization of the counting points formula for the Shimura varieties just mentioned (see [14] for the Drinfeld case).

This article overlaps with [11] in the case where $\chi = \mathrm{triv}$, for then $\mathcal{H}(G,\rho)$ is just the Iwahori-Hecke algebra and the theorem proved here is a special case of [11]. Here we use Labesse's method of elementary functions [24], whereas in [11] we followed Clozel's method [7] more closely. When $\chi = \mathrm{triv}$, many arguments presented here become simpler. Thus in that special case, this article gives an alternative (somewhat easier) proof for the Iwahori-Hecke algebra special case of [11].

As in earlier papers, the proof of Theorem 1.1 is by induction on the semisimple rank of G, and so consists of two steps: (i) use descent formulas to reduce to the case of elliptic elements, and (ii) use a global trace formula argument to prove the theorem for suitable elliptic elements. Labesse's elementary functions are used in step (ii) to give sufficiently many character identities that the required character identity for central elements is forced (see Section 9).

Here is an outline of the contents of the paper. In Section 2 we recall some standard notation which will be used throughout the article. In Section 3 we review the essential facts about the Bernstein decomposition related to depth-zero principal series blocks. In Section 4 we define the base change homomorphism and prove some of its basic properties. The constant term homomorphism (an essential ingredient for descent) is studied in Section 5, and in particular is proved to be compatible with the base change homomorphisms. The descent formulas themselves are the subject of Section 6. Section 7 reduces the general fundamental lemma to the case where G is adjoint and γ is a norm and is elliptic and strongly regular semisimple. (Unfortunately, this section is the most technical section of all.) In Section 8 we introduce and study suitable analogues of Labesse's elementary functions adapted to the Bernstein component $\mathcal{R}_\chi(G)$. In Section 9 we use all the preceding material to conclude the proof by establishing the character identity which is equivalent, by the existence of the local data, to the required identity of stable (twisted) orbital integrals. In Section 10 we define (1.0.2) and explain how Theorem 1.1 implies Corollary 1.2. Finally, in Section 11 we correct and clarify a few minor mistakes in [11].

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2. Further notation

Let L denote $\widehat{F^{un}}$, the completion of the maximal unramified extension of F contained in $\bar{F}$. Let F_r/F be an unramified extension of degree r contained in L, with ring of integers $\mathcal{O}_r$ and residue field k_r. Let $\sigma \in \mathrm{Aut}(L/F)$ denote the Frobenius automorphism, and use the same symbol to denote the induced automorphism on groups of the form $G(L)$, etc. Fix an algebraic closure $\bar{L}$ for L and define the inertia subgroup[(2)] as $I = \mathrm{Gal}(\bar{L}/L)$.

Denote groups of F_r-points with a subscript r, e.g. $T_r := T(F_r)$ and $G_r := G(F_r)$. Fix a generator $\theta \in \mathrm{Gal}(F_r/F)$. We use the same symbol θ to denote the induced automorphisms of groups of F_r-points T_r, G_r, etc.

Let ϖ denote a uniformizer for the field F.

We recall the basic facts on the Kottwitz homomorphism [23]. In loc. cit. is defined a canonical surjective homomorphism for any connective reductive F-group H

$$\kappa_H : H(L) \twoheadrightarrow X^*(Z(\widehat{H}))_I, \tag{2.0.3}$$

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(2) Not to be confused with the Iwahori subgroup!