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LINEAR RESPONSE FOR SMOOTH DEFORMATIONS OF GENERIC NONUNIFORMLY HYPERBOLIC UNIMODAL MAPS

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ABSTRACT. – We consider C^2 families $t \mapsto f_t$ of C^4 unimodal maps f_t whose critical point is slowly recurrent, and we show that the unique absolutely continuous invariant measure μ_t of f_t depends differentiably on t, as a distribution of order 1. The proof uses transfer operators on towers whose level boundaries are mollified via smooth cutoff functions, in order to avoid artificial discontinuities. We give a new representation of μ_t for a Benedicks-Carleson map f_t, in terms of a single smooth function and the inverse branches of f_t along the postcritical orbit. Along the way, we prove that the twisted cohomological equation $v = \alpha \circ f - f'\alpha$ has a continuous solution α, if f is Benedicks-Carleson and v is horizontal for f.

RÉSUMÉ. – Nous considérons des familles $t \mapsto f_t$ d'applications unimodales C^4, de récurrence postcritique lente, avec une dépendance C^2 en fonction du paramètre t. Nous montrons que l'unique mesure invariante μ_t de f_t est différentiable en fonction de t, en tant que distribution d'ordre 1. La preuve utilise des opérateurs de transfert sur des tours dont les bords sont mollifiés avec des fonctions de troncation lisses, pour éviter l'introduction de discontinuités artificielles. Nous donnons de plus une représentation de μ_t dépendant d'une unique fonction lisse et des branches inverses de f_t le long de l'orbite postcritique. Nous prouvons enfin que l'équation cohomologique tordue $v = \alpha \circ f - f'\alpha$ admet une solution continue α, si f est Benedicks-Carleson et v est horizontal pour f.

1. Introduction

The linear response problem for discrete-time dynamical systems can be posed in the following way. Suppose that for each parameter t (or many parameters t) in a smooth family

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of maps $t \mapsto f_t$ with $f_t\colon M \to M$, (M a compact Riemann manifold, say) there exists a unique physical (or SRB) measure μ_t. (See [63] for a discussion of SRB measures.) One can ask for conditions which ensure the differentiability, possibly in the sense of Whitney, of the function μ_t in a weak sense (in the weak $*$-topology, i.e., as a distribution of order 0, or possibly as a distribution of higher order). Ruelle has discussed this problem in several survey papers [46], [48], [50], to which we refer for motivation.

The case of smooth hyperbolic dynamics has been settled over a decade ago ([25], [45]), although recent technical progress in the functional analytic tools (namely, the introduction of anisotropic Sobolev spaces on which the transfer operator has a spectral gap) has allowed for a great simplification of the proofs (see, e.g., [19]): For smooth Anosov diffeomorphisms f_s and a C^1 observable A, letting

$$X_s = \partial_t f_t|_{t=s} \circ f_s^{-1}\,,$$

Ruelle [45], [47] obtained the following explicit *linear response formula* (the derivative here is in the usual sense)

$$\partial_t \int A\, d\mu_t|_{t=0} = \Psi_A(1)\,,$$

where $\Psi_A(z)$ is the *susceptibility function*

$$\Psi_A(z) = \sum_{k=0}^{\infty} \int z^k \langle X_0, \operatorname{grad}(A \circ f_0^k)\rangle\, d\mu_0\,,$$

and the series $\Psi_A(z)$ at $z = 1$ converges exponentially. In fact, in the Anosov case, the susceptibility function is holomorphic in a disc of radius larger than 1. This is related to the fact (see [7] for a survey and references) that the transfer operator of each f_s has a spectral gap on a space which contains not only the product of the distribution μ_s and the smooth vector field X_s, but also the derivative of that product, that is, $\langle X_s, \operatorname{grad}\mu_s\rangle + (\operatorname{div} X_s)\,\mu_s$.

One feature of smooth hyperbolic dynamics is structural stability: Each f_t, for small t, is topologically conjugated to f_0 via a homeomorphism h_t, which turns out to depend smoothly on the parameter t. With the exception of a deep result of Dolgopyat [21] on rapidly mixing partially hyperbolic systems (where structural stability may be violated, but where there are no critical points and shadowing holds for a set of points of large measure, so that the bifurcation structure is relatively mild), the study of linear response in the absence of structural stability, or in the presence of critical points, has begun only recently.

However, the easier property of *continuity* of μ_t with respect to t (in other words, *statistical stability*) has been established also in the presence of critical points: For piecewise expanding unimodal interval maps, Keller [26] proved in 1982 that the density ϕ_t of μ_t, viewed as an element of L^1, has a modulus of continuity at least $t \ln t$, so that $t \mapsto \phi_t$ is r-Hölder, for any exponent $r \in (0,1)$. For nonuniformly smooth unimodal maps, in general not all nearby maps f_t admit an SRB measure even if f_0 does. Therefore, continuity of $t \mapsto \mu_t$ can only be proved in the sense of Whitney, on a set of "good" parameters. This was done by Tsujii [58] and Rychlik–Sorets [53] in the 90's. More recently, Alves et al. [2], [1] proved that for Hénon maps, $t \mapsto \mu_t$ is continuous in the sense of Whitney in the weak $*$-topology. (We refer, e.g., to [8] for more references.)

Differentiability of μ_t, even in the sense of Whitney, is a more delicate issue, even in dimension one. For nonuniformly hyperbolic smooth unimodal maps f_t with a quadratic critical point ($f_t''(c) < 0$), it is known [61], [30] that the density ϕ_t of the absolutely continuous invariant measure μ_t of f_t has singularities (called *spikes*) of the form $\sqrt{x - c_{k,t}}^{-1}$, where the $c_{k,t} = f_t^k(c)$ are the points along the forward orbit of the critical point c. Thus, the derivative ϕ_t' of the invariant density has nonintegrable singularities, and the transfer operator cannot have a spectral gap in general on a space containing $(X_t\phi_t)'$. In fact, the radius of convergence of the susceptibility function $\Psi_A(z)$ is very likely strictly smaller than 1 in general. Ruelle [49] observed however that, in the case of a subhyperbolic (preperiodic) critical point for a real analytic unimodal map, $\Psi_A(z)$ is meromorphic in a disc of radius larger than 1, and that 1 is not a pole of $\Psi_A(z)$. He expressed the hope that the value $\Psi_A(1)$ obtained by analytic continuation could correspond to the actual derivative of the SRB measure, at least in the sense of Whitney.

This analytic continuation phenomenon in the subhyperbolic smooth unimodal case (where a finite Markov partition exists) could well be a red herring, in view of the linear response theory for the "toy model" of piecewise expanding interval maps that we recently established in a series of papers [7], [10], [12], [13]: Unimodal piecewise expanding interval maps f_t have a unique SRB measure, whose density ϕ_t is a function of bounded variation (since ϕ_t' is a measure, the situation is much easier than for smooth unimodal maps). In [7], [10], [14], we showed that Keller's [26] $t \ln t$ modulus of continuity was optimal (see also [35]): In fact, there exist smooth families f_t so that $t \mapsto \mu_t$ is not Lipschitz, even when viewed as a distribution of arbitrarily high order, and even in the sense of Whitney. Such counter-examples f_t are transversal to the topological class of f_0. If, on the contrary, the family f_t is tangent at $t = 0$ to the topological class of f_0 (we say that f_t is *horizontal*) then ([10], [12]) we proved that the map $t \mapsto \mu_t$ is differentiable for the weak $*$-topology. The series for $\Psi_A(1)$ may diverge (for the preperiodic case, see [7, §5]), but can be resumed under the horizontality condition [7], [10]. This gives an explicit linear response formula. In fact, the susceptibility function $\Psi_A(z)$ is holomorphic in the open unit disc, and, under a condition slightly stronger than horizontality, $\partial_t \int A\, d\mu_t|_{t=0}$ is the Abel limit of $\Psi_A(z)$ as $z \to 1$.

Worrying about lack of differentiability of the SRB measure is not just a mathematician's pedantry: Indeed, this phenomenon can be observed numerically, for example in the guise of fractal transport coefficients. We refer, e.g., to the work of Keller et al. [28] (see also references therein), who obtained a $t\ln(t)$ modulus of continuity compatible with the results of [26], for drift and diffusion coefficients of models related to those analyzed in [10] [14].

Let us move on now to the topic of the present work, linear response for smooth unimodal interval maps: Ruelle recently obtained a linear response formula for real analytic families of analytic unimodal maps of Misiurewicz type [51], that is, assuming $\inf_k |f^k(c) - c| > 0$, a nongeneric condition which implies the existence of a hyperbolic Cantor set. (Again, this linear response formula can be viewed as a resumation of the generally divergent series $\Psi_A(1)$.) In [11], we showed that $t \mapsto \mu_t$ is real analytic in the weak sense for complex analytic families of Collet-Eckmann quadratic-like maps (the – very rigid – holomorphicity assumption allowed us to use tools from complex analysis). Both these recent results are for families f_t in the conjugacy class of a single (analytic) unimodal map, and the assumptions were somewhat nongeneric.

The main result of the present work, Theorem 2.13, is a linear response formula for C^2 families $t \mapsto f_t$ of C^4 unimodal maps [(1)] with quadratic critical points satisfying the so-called *topological slow recurrence* (TSR) condition ([54],[57],[32], see (5) below). (We assume that the maps have negative Schwarzian and are symmetric, to limit technicalities, and we only consider infinite postcritical orbits, since the preperiodic case is much easier.) The topological slow recurrence condition is much weaker than Misiurewicz, so that we give a new proof of Ruelle's result [51] in the symmetric infinite postcritical case (this may shed light on the informal study in §17 there). Topological slow recurrence implies the well-known Benedicks-Carleson and Collet-Eckmann conditions. Furthermore, the work of Tsujii [57] and Avila-Moreira [6] gives that real-analytic unimodal maps with a quadratic critical point satisfying the TSR condition are measure-theoretical *generic* among non regular parameter in non trivial real-analytic families unimodal maps. (See Remark 2.3.) If all maps in a family of unimodal maps f_t satisfy the topological slow recurrence condition then [55] this family is a *deformation,* that is, the family $\{f_t\}$ lies entirely in the topological class of f_0 (there exist homeomorphisms h_t such that $h_t(c) = c$ and $h_t \circ f_0 = f_t \circ h_t$). In particular, horizontality holds.

We next briefly discuss a few new ingredients of our arguments, as well as a couple of additional results we obtained along the way. A first remark is that we need uniformity of the hyperbolicity constants of f_t for all small t. We deduce this uniformity from previous work of Nowicki, making use of the TSR assumption (Section 5).

When one moves the parameter t, the orbit of the critical point also moves, and so do the spikes. Therefore, in order to understand $\partial_t \mu_t$, we need upper bounds on

$$\partial_t c_{k,t}|_{t=0} = \partial_t f_t^k(c)|_{t=0} = \partial_t h_t(f_0^k(c))|_{t=0} = \partial_t h_t(c_{k,0})|_{t=0}\,,$$

uniformly in k. It is not very difficult to show (Lemma 2.10, see also Proposition 2.15) that $\partial_t c_{k,t}|_{t=0} = \alpha(c_{k,0})$ if α solves the twisted cohomological equation [(2)] (TCE) for $v = \partial_t f_t|_{t=0}$, given by,

$$v = \alpha \circ f_0 + f_0' \cdot \alpha\,, \quad \alpha(c) = 0\,.$$

(Such a function α is called an *infinitesimal conjugacy.)* In fact, we prove in Theorem 2.4 that if f_0 is Benedicks-Carleson and v satisfies a horizontality condition for f_0, then the TCE above has a unique solution α. In addition, α is continuous.

In the case of piecewise expanding maps on the interval, the invariant density ϕ_t is a fixed point of a Perron-Frobenius type transfer operator $\mathcal{L}_t$ in an appropriate space, where 1 is a simple isolated eigenvalue. So if we are able to verify some (weak) smoothness in the family $t \to \mathcal{L}_t$, then we can show (weak) differentiability of μ_t by using perturbation theory. (We may use different norms in the range and the domain, in the spirit of Lasota-Yorke or Doeblin-Fortet inequalities.) This is, roughly speaking, what was done in [10] and [13] (as already mentioned, a serious additional difficulty in the presence of critical points, which had to be overcome even in the toy model, is the absence of a spectral gap on a space containing the derivative of the invariant density). For Collet-Eckmann unimodal maps f_t, however, an

(1) The C^4 regularity is only used to get W_1^2 regularity in Proposition 4.11 and Lemma 4.12, and one can perhaps weaken this to $C^{3+\eta}$.

(2) In one-dimensional dynamics, the acronym TCE also stands [44] for "topological Collet-Eckmann," there should be no confusion since the topological Collet-Eckmann condition is not used here.