Ann. Scient. Éc. Norm. Sup.
4e série, t. 46, 2013, p. 129 à 173

RANDOM WALKS ON CO-COMPACT FUCHSIAN GROUPS

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ABSTRACT. – It is proved that the Green's function of a symmetric finite range random walk on a co-compact Fuchsian group decays exponentially in distance at the radius of convergence R. It is also shown that Ancona's inequalities extend to R, and therefore that the Martin boundary for R-potentials coincides with the natural geometric boundary S^1, and that the Martin kernel is uniformly Hölder continuous. Finally, this implies a local limit theorem for the transition probabilities: in the aperiodic case, $p^n(x, y) \sim C_{x,y} R^{-n} n^{-3/2}$.

RÉSUMÉ. – Considérons une marche aléatoire symétrique à support fini sur un groupe fuchsien co-compact. Nous montrons que la fonction de Green à son rayon de convergence R décroît exponentiellement vite en fonction de la distance à l'origine. Nous montrons également que les inégalités d'Ancona s'étendent jusqu'au paramètre R, et par conséquent que la frontière de Martin pour les R-potentiels s'identifie avec la frontière géométrique S^1. De plus, le noyau de Martin correspondant est höldérien. Ces résultats sont utilisés pour démontrer un théorème limite local pour les probabilités de transition : dans le cas apériodique, $p^n(x, y) \sim C_{x,y} R^{-n} n^{-3/2}$.

1. Introduction

1.1. Green's function and Martin boundary

A (right) *random walk* on a countable group Γ is a discrete-time Markov chain $\{X_n\}_{n\geq 0}$ of the form

$$X_n = x\xi_1\xi_2 \cdots \xi_n$$

where $\xi_1, \xi_2, \ldots$ are independent, identically distributed Γ-valued random variables. The distribution of ξ_i is the *step distribution* of the random walk. The random walk is said to be *symmetric* if its step distribution is invariant under the mapping $x \mapsto x^{-1}$, and *finite-range* if the step distribution has finite support. The *Green's function* is the generating function of the

Second author supported by NSF grant DMS - 0805755.

transition probabilities: for $x, y \in \Gamma$ and $0 \leq r < 1$ it is defined by the absolutely convergent series

$$G_r(x,y) := \sum_{n=0}^{\infty} P^x\{X_n = y\} r^n = G_r(1, x^{-1}y); \tag{1}$$

here P^x is the probability measure on path space governing the random walk with initial point x. If the random walk is irreducible (that is, if the semigroup generated by the support of the step distribution is Γ) then the radius of convergence R of the series (1) is the same for all pairs x, y. Moreover, if the random walk is symmetric, then $1/R$ is the *spectral radius* of the transition operator. By a fundamental theorem of Kesten [25], if the group Γ is finitely generated and nonamenable then $R > 1$. Moreover, in this case the Green's function is finite at its radius of convergence (cf. [39], ch. 2): for all $x, y \in \Gamma$,

$$G_R(x,y) < \infty. \tag{2}$$

The Green's function is of central importance in the study of random walks. Clearly, it encapsulates information about the transition probabilities; in Theorem 9.1, we show that the local asymptotic behavior of the transition probabilities can be deduced from the singular behavior of the Green's function at its radius of convergence. The Green's function is also the key to the potential theory associated with the random walk: in particular, it determines the Martin boundary for r-potential theory. A prominent theme in the study of random walks on nonabelian groups has been the relationship between the geometry of the group and the nature of the Martin boundary. A landmark result here is a theorem of Ancona [2] describing the Martin boundary for random walks with finitely supported step distributions on *hyperbolic* groups: Ancona proves that for every $r \in (0, R)$ the Martin boundary for r-potential theory coincides with the *geometric* (Gromov) boundary, in a sense made precise below. (Series [35] had earlier established this in the special case $r = 1$ when the group is co-compact Fuchsian. See also [3] and [1] for related results concerning Laplace-Beltrami operators on Cartan manifolds.)

It is natural to ask whether Ancona's theorem extends to $r = R$, that is, if the Martin boundary is stable (see [33] for the terminology) through the entire range $(0, R]$. One of the main results of this paper (Theorem 1.3) provides an affirmative answer in the special case of symmetric, finite-range random walk on a co-compact Fuchsian group, i.e., a co-compact, discrete subgroup of $PSL(2, \mathbb{R})$. Any co-compact Fuchsian group acts as a discrete group of isometries of the hyperbolic disk, and so its Cayley graph can be embedded quasi-isometrically in the hyperbolic disk; this implies that its Gromov boundary is the circle S^1 at infinity.

THEOREM 1.1. – *For any symmetric, irreducible, finite-range random walk on a co-compact Fuchsian group Γ, the Martin boundary for R-potentials coincides with the geometric boundary $S^1 = \partial\Gamma$. Moreover, all elements of the Martin boundary are minimal.*

This assertion means that (a) for every geodesic ray $y_0, y_1, y_2, \dots$ in the Cayley graph that converges to a point $\zeta \in \partial\Gamma$ and for every $x \in \Gamma$,

$$\lim_{n\to\infty} \frac{G_R(x, y_n)}{G_R(1, y_n)} = K_R(x, \zeta) = K(x, \zeta) \tag{3}$$

exists; (b) for each $\zeta \in \partial\Gamma$ the function $K_\zeta(x) := K(x, \zeta)$ is a minimal, positive R-harmonic function of x; (c) for distinct points $\zeta, \zeta' \in \partial\Gamma$ the functions K_ζ and $K_{\zeta'}$ are different; and (d) the topology of pointwise convergence on $\{K_\zeta\}_{\zeta \in \partial\Gamma}$ coincides with the usual topology on $\partial\Gamma = S^1$.

Our results also yield explicit rates for the convergence (3), and imply that the Martin kernel $K_r(x, \zeta)$ is *Hölder* continuous in ζ relative to the usual Euclidean metric (or any visual metric—see [23] for the definition) on $S^1 = \partial\Gamma$.

THEOREM 1.2. – *For any symmetric, irreducible, finite-range random walk on a co-compact Fuchsian group* Γ*, there exists* $\varrho < 1$ *such that for every* $1 \le r \le R$ *and every geodesic ray* $1 = y_0, y_1, y_2, \ldots$ *converging to a point* $\zeta \in \partial\Gamma$*,*

$$\left| \frac{G_r(x, y_n)}{G_r(1, y_n)} - K_r(x, \zeta) \right| \le C_x \varrho^n. \tag{4}$$

The constants $C_x < \infty$ *depend on* $x \in \Gamma$ *but not on* $r \le R$*. Consequently, for each* $x \in \Gamma$ *and* $r \le R$ *the function* $\zeta \mapsto K_r(x, \zeta)$ *is Hölder continuous in* ζ *relative to the Euclidean metric on* $S^1 = \partial\Gamma$*, for some exponent not depending on* $r \le R$*.*

The exponential convergence (4) and the Hölder continuity of the Martin kernel for $r = 1$ were established by Series [34] for random walks on Fuchsian groups. Similar results for the Laplace-Beltrami operator on negatively curved Cartan manifolds were proved by Anderson and Schoen [3]. The methods of [3] were adapted by Ledrappier [30] to prove that Series' results extend to all random walks on a free group, and Ledrappier's proof was extended by Izumi, Neshvaev, and Okayasu [22] to prove that for a random walk on a non-elementary hyperbolic group the Martin kernel $K_1(x, \zeta)$ is Hölder continuous in ζ. All of these proofs rest on inequalities of the type discussed in Section 1.2 below. Theorem 1.3 below asserts (among other things) that similar estimates are valid for all G_r *uniformly for* $r \le R$. Given these, the proof of [22] applies almost verbatim to establish Theorem 1.2. We will give some additional details in Paragraph 4.2.

1.2. Ancona's inequalities

The crux of Ancona's argument in [2] was a system of inequalities that assert, roughly, that the Green's function $G_r(x, y)$ is nearly submultiplicative in the arguments $x, y \in \Gamma$. Ancona [2] proved that such inequalities always hold for $r < R$: in particular, he proved, for a random walk with finitely supported step distribution on a hyperbolic group, that for each $r < R$ there is a constant $C_r < \infty$ such that for every geodesic segment $x_0 x_1 \cdots x_m$ in (the Cayley graph of) Γ,

$$G_r(x_0, x_m) \le C_r G_r(x_0, x_k) G_r(x_k, x_m) \qquad \forall\, 1 \le k \le m. \tag{5}$$

His argument depends in an essential way on the hypothesis $r < R$ (cf. his Condition (*)), and it leaves open the possibility that the constants C_r in the inequality (5) might blow up as $r \to R$. For finite-range random walk on a *free* group it can be shown, by direct calculation, that the constants C_r remain bounded as $r \to R$, and that the inequalities (5) remain valid at $r = R$ (cf. [27]). The following result asserts that the same is true for symmetric random walks on a co-compact Fuchsian group.

THEOREM 1.3. – *For any symmetric, irreducible, finite-range random walk on a co-compact Fuchsian group* Γ*,*

(A) *the Green's function* $G_R(1,x)$ *decays exponentially in* $|x| := d(1,x)$*; and*
(B) *Ancona's inequalities* (5) *hold for all* $r \le R$*, with a constant* C *independent of* r*.*

NOTE 1.4. – Here and throughout the paper $d(x,y)$ denotes the distance between the vertices x and y in the Cayley graph G^Γ, equivalently, distance in the word metric. *Exponential decay* of the Green's function means *uniform* exponential decay in all directions, that is, there are constants $C < \infty$ and $\varrho < 1$ such that for all $x, y \in \Gamma$,

$$G_R(x,y) \le C\varrho^{d(x,y)}. \tag{6}$$

A very simple argument (see Lemma 2.1 below) shows that for a symmetric random walk on any nonamenable group $G_R(1,x) \to 0$ as $|x| \to \infty$. Given this, it is routine to show that exponential decay of the Green's function follows from Ancona's inequalities. However, we will argue in the other direction, first providing an independent proof of exponential decay in Subsection 3.3, and then deducing Ancona's inequalities from it in Section 4.

NOTE 1.5. – Theorem 1.3 (A) is a discrete analogue of one of the main results (Theorem B) of Hamenstaedt [21] concerning the Green's function of the Laplacian on the universal cover of a compact negatively curved manifold. Unfortunately, Hamenstaedt's proof appears to have a serious error.(1) The approach taken here bears no resemblance to that of [21].

Theorem 1.3 is proved in Sections 3 and 4 below. The argument uses the *planarity* of the Cayley graph in an essential way. It also relies on the simple estimate

$$\lim_{|x|\to\infty} G_R(1,x) = 0,$$

that we derive from the symmetry of the random walk. While this estimate is not true in general without the symmetry assumption, we nevertheless conjecture that Ancona's inequalities and the identification of the Martin boundary at $r = R$ hold in general.

1.3. Decay at infinity of the Green's function

Neither Ancona's result nor Theorem 1.3 gives any information about how the uniform exponential decay rate ϱ depends on the step distribution of the random walk. In fact, the Green's function $G_r(1,x)$ decays at different rates in different directions $x \to \partial\Gamma$. To quantify the overall decay, consider the behavior of the Green's function over the entire sphere S_m of radius m centered at 1 in the Cayley graph G^Γ. If Γ is a nonelementary Fuchsian group then the cardinality of the sphere S_m grows exponentially in m (see Corollary 5.5 in Section 5), that is, there exist constants $C > 0$ and $\zeta > 1$ such that as $m \to \infty$,

$$|S_m| \sim C\zeta^m.$$

(1) The error is in the proof of Lemma 3.1: The claim is made that a lower bound on a finite measure implies a lower bound for its Hausdorff-Billingsley dimension relative to another measure. This is false—in fact such a lower bound on measure implies an *upper* bound on its Hausdorff-Billingsley dimension.

THEOREM 1.6. – *For any symmetric, irreducible, finite-range random walk on a co-compact Fuchsian group* Γ*,*

$$\lim_{m\to\infty} \sum_{x\in S_m} G_R(1,x)^2 = C > 0 \tag{7}$$

exists and is finite, and

$$\#\{x \in \Gamma \,:\, G_R(1,x) \geq \varepsilon\} \asymp \varepsilon^{-2} \tag{8}$$

as $\varepsilon \to 0$*. (Here* $\asymp$ *means that the ratio of the two sides remains bounded away from* 0 *and* ∞*.)*

The proof is carried out in Sections 6–7 below (cf. Propositions 6.2 and 7.1), using the fact that any hyperbolic group has an *automatic structure* [19]. The automatic structure will permit us to use the theory of *Gibbs states* and *thermodynamic formalism* of Bowen [11], ch. 1. Theorem 1.2 is essential for this, as the theory developed in [11] applies only to Hölder continuous functions.

It is likely that $\asymp$ can be replaced by $\sim$ in (8). There is a simple heuristic argument that suggests why the sums $\sum_{x\in S_m} G_R(1,x)^2$ should remain bounded as $m \to \infty$: Since the random walk is R-transient, the contribution to $G_R(1,1) < \infty$ from random walk paths that visit S_m and then return to 1 is bounded (by $G_R(1,1)$). For any $x \in S_m$, the term $G_R(1,x)^2/G_R(1,1)$ is the contribution to $G_R(1,1)$ from paths that visit x before returning to 1. Thus, if $G_R(1,x)$ is not substantially larger than

$$\sum_{n=1}^{\infty} P^1\{X_n = x \text{ and } \tau(m) = n\}R^n,$$

where $\tau(m)$ is the time of the first visit to S_m, then the sum in (7) should be of the same order of magnitude as the total contribution to $G_R(1,1) < \infty$ from random walk paths that visit S_m and then return to 1. Of course, the difficulty in making this heuristic argument rigorous is that *a priori* one does not know that paths that visit x are likely to be making their first visits to S_m; it is Ancona's inequality (5) that ultimately fills the gap.

NOTE 1.7. – A simple argument shows that for $r > 1$ the sum of the Green's function on the sphere S_m, unlike the sum of its square, explodes as $m \to \infty$. Fix $1 < r \leq R$ and $m \geq 1$. Let C_0 bound the size of the jumps of the random walk, and let $\tilde{S}_m$ be the set of points with $d(1,x) \in [m, m + C_0)$. Since X_n is transient, it will, with probability one, eventually visit the annulus $\tilde{S}_m$. The minimum number of steps needed to reach $\tilde{S}_m$ is at least m/C_0. Hence,

$$\begin{aligned}
\sum_{x\in\tilde{S}_m} G_r(1,x) &= \sum_{n=m/C_0}^{\infty} \sum_{x\in\tilde{S}_m} P^1\{X_n = x\}r^n \\
&\geq r^{m/C_0} \sum_{n=m/C}^{\infty} P^1\{X_n \in \tilde{S}_m\} \\
&\geq r^{m/C_0} P^1\{X_n \in \tilde{S}_m \text{ for some } n\} \\
&= r^{m/C_0}.
\end{aligned}$$

Hence, $\sum_{x\in\tilde{S}_m} G_r(1,x)$ diverges. The divergence of $\sum_{x\in S_m} G_r(1,x)$ readily follows if the random walk is irreducible.