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space dimension*

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TRAVELLING GRAPHS FOR THE FORCED MEAN CURVATURE MOTION IN AN ARBITRARY SPACE DIMENSION

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Dedicated to Henri Berestycki

ABSTRACT. – We construct travelling wave graphs of the form $z = -ct + \phi(x)$, $\phi : x \in \mathbb{R}^{N-1} \mapsto \phi(x) \in \mathbb{R}$, $N \geq 2$, solutions to the N -dimensional forced mean curvature motion $V_n = -c_0 + \kappa$ ($c \geq c_0$) with prescribed asymptotics. For any 1-homogeneous function ϕ_∞ , viscosity solution to the eikonal equation $|D\phi_\infty| = \sqrt{(c/c_0)^2 - 1}$, we exhibit a smooth concave solution to the forced mean curvature motion whose asymptotics is driven by ϕ_∞ . We also describe ϕ_∞ in terms of a probability measure on \mathbb{S}^{N-2} .

RÉSUMÉ. – Nous construisons des ondes progressives sous la forme de graphes $z = -ct + \phi(x)$, $\phi : x \in \mathbb{R}^{N-1} \mapsto \phi(x) \in \mathbb{R}$, $N \geq 2$, solutions du mouvement par courbure moyenne forcée $V_n = -c_0 + \kappa$ ($c \geq c_0$) en dimension N d'espace et avec un comportement asymptotique prescrit. Pour toute solution de viscosité ϕ_∞ , 1-homogène en espace, de l'équation eikonale $|D\phi_\infty| = \sqrt{(c/c_0)^2 - 1}$, nous mettons en évidence une solution régulière et concave du mouvement par courbure moyenne forcée dont le comportement asymptotique est donné par ϕ_∞ . Nous décrivons aussi ϕ_∞ en terme d'une mesure de probabilité sur la sphère \mathbb{S}^{N-2} .

1. Introduction

1.1. Setting of the problem

The question investigated here is the description of the travelling wave graph solutions to the forced mean curvature motion in any dimension $N \geq 2$, that is written under the general form

$$(1) \quad V_n = -c_0 + \kappa$$

where V_n is the normal velocity of the graph, κ its local mean curvature and c_0 a given strictly positive constant to be defined later. A graph satisfying (1) can be given by the equation

$z = u(t, x)$ where $u : (t, x) \in \mathbb{R}^+ \times \mathbb{R}^{N-1} \mapsto u(t, x) \in \mathbb{R}$ is a solution to the parabolic equation

$$(2) \quad \frac{u_t}{\sqrt{1 + |Du|^2}} = -c_0 + \operatorname{div} \left(\frac{Du}{\sqrt{1 + |Du|^2}} \right), \quad t > 0, x \in \mathbb{R}^{N-1}.$$

Indeed, at any time $t > 0$ fixed, the outer normal to the subgraph $\{(x, z) \in \mathbb{R}^{N-1} \times \mathbb{R} \mid z \leq u(t, x)\}$ is given by

$$\vec{n} = \frac{1}{\sqrt{1 + |Du|^2}} \begin{pmatrix} -D_x u \\ 1 \end{pmatrix}$$

its normal velocity V_n by $(0, \partial_t u)^T \cdot \vec{n}$ while its mean curvature by $\kappa = -\operatorname{div}_{(x,z)} \vec{n}$, see [6].

A travelling wave to (2) is a solution of the form $u(t, x) = -ct + \phi(x)$ where $\phi : x \in \mathbb{R}^{N-1} \mapsto \phi(x) \in \mathbb{R}$ is the profile of the wave and $c \geq c_0$ is some given constant standing for its speed. Thus ϕ satisfies the following elliptic equation

$$(3) \quad -\operatorname{div} \left(\frac{D\phi}{\sqrt{1 + |D\phi|^2}} \right) + c_0 - \frac{c}{\sqrt{1 + |D\phi|^2}} = 0, \quad x \in \mathbb{R}^{N-1}.$$

1.2. Connection with reaction diffusion equations

This work should provide us a better understanding of the multidimensional solutions to the non linear scalar reaction diffusion equation

$$(4) \quad \partial_t v = \Delta v + f(v), \quad t > 0, (x, z) \in \mathbb{R}^{N-1} \times \mathbb{R}$$

where $v : (t, x, z) \in [0, +\infty) \times \mathbb{R}^{N-1} \times \mathbb{R} \mapsto v(t, x, z) \in \mathbb{R}$ and, especially the case of travelling waves in dimension N . In the case of a “bistable” nonlinearity f , that is to say when f is a continuously differentiable function on \mathbb{R} satisfying

- (i) $f(0) = f(1) = 0$
- (ii) $f'(0) < 0$ and $f'(1) < 0$
- (iii) there exists $\theta \in (0, 1)$ such that $f(v) < 0$ for $v \in (0, \theta)$, $f(v) > 0$ for $v \in (\theta, 1)$
- (iv) $\int_0^1 f(v) dv > 0$,

it is well-known [10] that there exists a one-dimensional travelling front $v(t, z) = \phi_0(z + c_0 t)$ solution to (4) with $N = 1$. The speed c_0 is unique and strictly positive by (iv) while the profile ϕ_0 is unique up to translations. This result defines the constant $c_0 > 0$ that appears in Equation (1).

In the case $N = 2$, multidimensional solutions to (4) are well understood. Paper [7] proves the existence of conical travelling waves solutions to (4), and paper [8] classifies all possible bounded non constant travelling waves solutions under rather weak conditions at infinity. In particular, it is proved in [8] that $c \geq c_0$ and, up to a shift in $x \in \mathbb{R}$, either u is a planar front $\phi_0(\pm x \cos \alpha + z \sin \alpha)$ with $\alpha = \arcsin(c_0/c) \in (0, \frac{\pi}{2}]$ or u is the unique conical front found in [7].

In higher dimensions, less is known. In [7], Hamel, Monneau and Roquejoffre or in [13], Ninomiya and Taniguchi proved the existence of conical travelling waves with cylindrical symmetry whose level sets are Lipschitz graphs moving away logarithmically from straight cones. Some special, non cylindrically symmetric pyramidal-shaped solutions (see Taniguchi [14] and references therein) are also known in any dimension $N \geq 3$.

Thus, in order to get a better understanding of the mechanisms at work, we further the idea of bridging reaction-diffusion equations with geometric motions. In particular, travelling wave graph solutions to the forced mean curvature motion go back to Fife [5]. He proved (in a formal fashion) that reaction-diffusion travelling fronts propagate with normal velocity

$$V_n = -c_0 + \frac{\kappa}{t} + O\left(\frac{1}{t^2}\right), \quad t \gg 1.$$

For a mathematically rigorous treatment of these ideas, we refer for instance to de Mottoni, Schatzman [11]—small times, smooth solutions context—and Barles, Soner, Souganidis [1]—arbitrary large times, viscosity solutions context.

1.3. Main results

Our Theorem 1.1 below states that, given a 1-homogeneous solution ϕ_∞ to the eikonal equation derived from (3) (i.e., the equation obtained by removing the curvature term) there exists a smooth solution ϕ to the forced mean curvature motion Equation (3) whose asymptotic behaviour is prescribed by ϕ_∞ . Here is the precise result.

Theorem 1.1 (Existence of solutions with prescribed asymptotics in dimension N)

Let $N \in \mathbb{N} \setminus \{0, 1\}$, $\alpha \in (0, \frac{\pi}{2}]$, $c_0 > 0$ and $c = c_0 / \sin \alpha$. Choose ϕ_∞ a 1-homogeneous viscosity solution to the eikonal equation

$$(5) \quad |D\phi_\infty(x)| = \cot \alpha, \quad x \in \mathbb{R}^{N-1}.$$

Then there exists a smooth concave solution $\phi \in C^\infty(\mathbb{R}^{N-1})$ to (3) such that

$$(6) \quad \phi(x) = \phi_\infty(x) + o(|x|) \text{ as } |x| \rightarrow +\infty.$$

This is the most possible general result. However, due to the possible complexity of a solution to the eikonal Equation (5), it is useful to specialize our result to the particular case of a solution with a finite number of facets.

Theorem 1.2 (Solutions with finite number of facets in dimension N)

Let $N \in \mathbb{N} \setminus \{0, 1\}$, $\alpha \in (0, \frac{\pi}{2}]$, $c_0 > 0$ and $c = c_0 / \sin \alpha$. Choose ϕ^* a viscosity solution to the eikonal Equation (5) given for any $x \in \mathbb{R}^{N-1}$ by

$$(7) \quad \phi^*(x) = \inf_{\nu \in A} (-(\cot \alpha) x \cdot \nu + \gamma_\nu)$$

where A is a finite subset of cardinal $k \in \mathbb{N}^*$ of the sphere \mathbb{S}^{N-2} and γ_ν are given real numbers.

Then there exists a unique smooth concave solution $\phi \in C^\infty(\mathbb{R}^{N-1})$ to (3) such that

$$(8) \quad \begin{cases} -\frac{2 \ln k}{c_0 \sin \alpha} \leq \phi - \phi^* \leq 0, & x \in \mathbb{R}^{N-1} \\ \lim_{l \rightarrow +\infty} \sup_{\text{dist}(x, E_\infty) \geq l} |\phi(x) - \phi^*(x)| = 0 \end{cases}$$

where E_∞ is the set of edges defined as

$$E_\infty = \{x \in \mathbb{R}^{N-1} \mid \phi_\infty \text{ is not } C^1 \text{ at } x\}$$

with the 1-homogeneous function

$$\phi_\infty(x) = \inf_{\nu \in A} (-(\cot \alpha) x \cdot \nu).$$

In space dimension $N = 3$, we obtain a more precise result by considering solutions having a finite number of gradient jumps. Those solutions are still more complex than the infimum of a finite number of affine forms. Here is the precise result.

Theorem 1.3 (Solutions with finite number of gradient jumps and $N = 3$)

Let $\alpha \in (0, \frac{\pi}{2}]$, $c_0 > 0$ and $c = c_0 / \sin \alpha$. Choose ϕ_∞ a 1-homogeneous viscosity solution to the eikonal Equation (5) in dimension $N = 3$ with a finite number of singularities on \mathbb{S}^1 . Then, there exist

- (1) a 2π -periodic continuous function $\psi_\infty : \theta \in [0, 2\pi] \mapsto \psi_\infty(\theta) \in [-\cot \alpha, \cot \alpha]$ and a finite number $k \in \mathbb{N} \setminus \{0\}$ of angles $\theta_1 < \dots < \theta_k$ in $[0, 2\pi)$ such that

$$\phi_\infty(r \cos \theta, r \sin \theta) = r\psi_\infty(\theta), \quad (r, \theta) \in \mathbb{R}^+ \times [0, 2\pi).$$

Moreover, for any $i \in \{1, \dots, k\}$,

- (a) Either $\forall \theta \in [\theta_i, \theta_{i+1}]$, $\psi_\infty(\theta) = -(\cot \alpha)$ and we set $\sigma_i = 1$.

(b) Or

$$\begin{cases} \forall \theta \in \left[\theta_i, \frac{\theta_i + \theta_{i+1}}{2} \right], & \psi_\infty(\theta) = -(\cot \alpha) \cos(\theta - \theta_i) \\ \forall \theta \in \left[\frac{\theta_i + \theta_{i+1}}{2}, \theta_{i+1} \right], & \psi_\infty(\theta) = -(\cot \alpha) \cos(\theta - \theta_{i+1}) \end{cases} \text{ and we set } \sigma_i = 0.$$

By convention, $\theta_{k+1} = 2\pi + \theta_1$ and $\sigma_{k+1} = \sigma_1$. If $k \geq 2$, then $\sigma_i \sigma_{i+1} = 0$ for any $i \in \{1, \dots, k\}$.

- (2) a smooth concave function $\phi \in C^\infty(\mathbb{R}^2)$ solution to Equation (3) such that when $|x|$ goes to infinity

$$\phi(x) = \phi_*(x) + O(1)$$

where

$$(9) \quad \phi_*(x) = -\frac{2}{c_0 \sin \alpha} \ln \left(\int_{\mathbb{S}^1} e^{\frac{c_0 \cos \alpha}{2} x \cdot \nu} d\mu(\nu) \right)$$

and μ is the non negative measure on \mathbb{S}^1 with finite mass determined by ψ_∞ as follows:

We set $\mu = \sum_{i=1}^k \mu_i$ where for any fixed $\lambda_0 > 0$, we set

- (a) If $\sigma_i = 1$, then $\mu_i = \mathbb{1}_{(\theta_i, \theta_{i+1})} d\theta + \lambda_0(\delta_{\theta_i} + \delta_{\theta_{i+1}})$
(with the exception for $k = 1$: $\mu_1 = \mathbb{1}_{(\theta_1, \theta_1 + 2\pi)} d\theta$).
- (b) If $\sigma_i = 0$, then $\mu_i = \lambda_0(\delta_{\theta_i} + \delta_{\theta_{i+1}})$.

We plan to use our travelling graphs for the forced mean curvature motion exhibited in Theorems 1.1 to 1.3 in order to construct multi-dimensional travelling fronts to the reaction diffusion Equation (4); we plan to do it in a forthcoming paper.

That Equation (5) prescribes the asymptotic behaviour of (3) has nothing surprising: let $\varepsilon > 0$ and denote by ϕ_ε the scaled function

$$\phi_\varepsilon(x) = \varepsilon \phi \left(\frac{x}{\varepsilon} \right), \quad x \in \mathbb{R}^{N-1}.$$

Since ϕ is a solution to (3), ϕ_ε satisfies

$$-\varepsilon \operatorname{div} \left(\frac{D\phi_\varepsilon}{\sqrt{1 + |D\phi_\varepsilon|^2}} \right) + c_0 - \frac{c}{\sqrt{1 + |D\phi_\varepsilon|^2}} = 0, \quad x \in \mathbb{R}^{N-1}.$$

Let ε go to zero. If adequate estimates for ϕ_ε are known, (a subsequence of) $(\phi_\varepsilon)_{\varepsilon > 0}$ converges to a function ϕ_∞ satisfying (5).