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KAM THEORY FOR THE HAMILTONIAN DERIVATIVE WAVE EQUATION

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ABSTRACT. – We prove an infinite dimensional KAM theorem which implies the existence of Cantor families of small-amplitude, reducible, elliptic, analytic, invariant tori of Hamiltonian derivative wave equations.

RÉSUMÉ. – Nous prouvons un théorème KAM en dimension infinie, qui implique l'existence de familles de Cantor de tores invariants de petite amplitude, réductibles, elliptiques et analytiques, pour les équations des ondes hamiltoniennes avec dérivées.

1. Introduction

In the last years many progresses have been done concerning KAM theory for nonlinear Hamiltonian PDEs. The first existence results were given by Kuksin [20] and Wayne [32] for semilinear wave (NLW) and Schrödinger equations (NLS) in one space dimension ($1d$) under Dirichlet boundary conditions, see [27]-[28] and [23] for further developments. The approach of these papers consists in generating iteratively a sequence of symplectic changes of variables which bring the Hamiltonian into a constant coefficients (=reducible) normal form with an elliptic (=linearly stable) invariant torus at the origin. Such a torus is filled by quasi-periodic solutions with zero Lyapunov exponents. This procedure requires to solve, at each step, constant-coefficients linear "homological equations" by imposing the "second order Melnikov" non-resonance conditions. Unfortunately these (infinitely many) conditions are violated already for periodic boundary conditions.

In this case, existence of quasi-periodic solutions for semilinear $1d$-NLW and NLS equations, was first proved by Bourgain [5] by extending the Newton approach introduced by Craig-Wayne [11] for periodic solutions. Its main advantage is to require only the "first order Melnikov" non-resonance conditions (the minimal assumptions) for solving the homological equations. Actually, developing this perspective, Bourgain was also able to prove in [6], [8] the existence of quasi-periodic solutions for NLW and NLS (with Fourier multipliers) in higher space dimensions; see also the recent extensions in [4], [3], [31]. The main drawback of this approach is that the homological equations are linear PDEs with non-constant coefficients. Translated in the KAM language this implies a non-reducible normal form around the torus and then a lack of informations about the stability of the quasi-periodic solutions.

Later on, existence of reducible elliptic tori was proved by Chierchia-You [9] for semilinear $1d$-NLW, and, more recently, by Eliasson-Kuksin [14] for NLS (with Fourier multipliers) in any space dimension; see also Procesi-Xu [30], Geng-Xu-You [15].

An important problem concerns the study of PDEs where the nonlinearity involves derivatives. A comprehension of this situation is of major importance since most of the models coming from Physics are of this kind.

In this direction KAM theory has been extended to deal with KdV equations by Kuksin [21]-[22], Kappeler-Pöschel [19], and, for the $1d$-derivative NLS (DNLS) and Benjiamin-Ono equations, by Liu-Yuan [24]. The key idea of these results is again to provide only a non-reducible normal form around the torus. However, in this case, the homological equations with non-constant coefficients are only *scalar* (not an infinite system as in the Craig-Wayne-Bourgain approach). We remark that the KAM proof is more delicate for DNLS and Benjiamin-Ono, because these equations are less "dispersive" than KdV, i.e., the eigenvalues of the principal part of the differential operator grow only quadratically at infinity, and not cubically as for KdV. As a consequence of this difficulty, the quasi-periodic solutions in [21], [19] are analytic, in [24], only C^∞. Actually, for the applicability of these KAM schemes, the more dispersive the equation is, the more derivatives in the nonlinearity can be supported. The limit case of the derivative nonlinear wave equation (DNLW)—which is not dispersive at all—is excluded by these approaches.

In the paper [5] (which proves the existence of quasi-periodic solutions for semilinear $1d$-NLS and NLW), Bourgain claims, in the last remark, that his analysis works also for the Hamiltonian "derivation" wave equation

$$y_{tt} - y_{xx} + g(x)y = \Big(- \frac{d^2}{dx^2}\Big)^{1/2} F(x,y);$$

see also [7], page 81. Unfortunately no details are given. However, Bourgain [7] provided a detailed proof of the existence of periodic solutions for the non-Hamiltonian equation

$$y_{tt} - y_{xx} + \mathrm{m}y + y_t^2 = 0\,, \quad \mathrm{m} \neq 0\,.$$

These kinds of problems have been then reconsidered by Craig in [10] for more general Hamiltonian derivative wave equations like

$$y_{tt} - y_{xx} + g(x)y + f(x, D^\beta y) = 0\,, \quad x \in \mathbb{T}\,,$$

where $g(x) \geq 0$ and D is the first order pseudo-differential operator $D := \sqrt{-\partial_{xx} + g(x)}$. The perturbative analysis of Craig-Wayne [11] for the search of periodic solutions works when $\beta < 1$. The main reason is that the wave equation vector field gains one derivative and then the nonlinear term $f(D^\beta u)$ has a strictly weaker effect on the dynamics for $\beta < 1$. The case $\beta = 1$ is left as an open problem. Actually, in this case, the small divisors problem for periodic solutions has the same level of difficulty of quasi-periodic solutions with 2 frequencies.

The goal of this paper is to extend KAM theory to deal with the Hamiltonian derivative wave equation

$$y_{tt} - y_{xx} + \mathrm{m}y + f(Dy) = 0\,, \quad \mathrm{m} > 0\,, \quad D := \sqrt{-\partial_{xx} + \mathrm{m}}\,, \quad x \in \mathbb{T}\,, \tag{1.1}$$

with real analytic nonlinearities (see Remark 7.1)

$$f(s) = as^3 + \sum_{k\geq 5} f_k s^k\,, \quad a \neq 0\,. \tag{1.2}$$

We write Equation (1.1) as the infinite dimensional Hamiltonian system

$$u_t = -\mathrm{i}\partial_{\bar u} H\,, \qquad \bar u_t = \mathrm{i}\partial_u H\,,$$

with Hamiltonian

$$H(u, \bar u) := \int_{\mathbb{T}} \bar u D u + F\Big(\frac{u + \bar u}{\sqrt 2}\Big)\, dx\,, \quad F(s) := \int_0^s f\,, \tag{1.3}$$

in the complex unknown

$$u := \frac{1}{\sqrt 2}(Dy + \mathrm{i}y_t)\,, \qquad \bar u := \frac{1}{\sqrt 2}(Dy - \mathrm{i}y_t)\,, \qquad \mathrm{i} := \sqrt{-1}\,.$$

Setting $u = \sum_{j\in\mathbb{Z}} u_j e^{\mathrm{i}jx}$ (similarly for $\bar u$), we obtain the Hamiltonian in infinitely many coordinates

$$H = \sum_{j\in\mathbb{Z}} \lambda_j u_j \bar u_j + \int_{\mathbb{T}} F\Big(\frac{1}{\sqrt 2}\sum_{j\in\mathbb{Z}} (u_j e^{\mathrm{i}jx} + \bar u_j e^{-\mathrm{i}jx})\Big)\, dx \tag{1.4}$$

where

$$\lambda_j := \sqrt{j^2 + \mathrm{m}} \tag{1.5}$$

are the eigenvalues of the diagonal operator D. Note that the nonlinearity in (1.1) is x-independent implying, for (1.3), the conservation of the momentum $-\mathrm{i}\int_{\mathbb{T}} \bar u \partial_x u\, dx$. This symmetry allows to simplify somehow the KAM proof (a similar idea was used by Geng-You [16]).

For every choice of the *tangential sites* $\mathcal{I} := \{\mathtt{j}_1, \dots, \mathtt{j}_n\} \subset \mathbb{Z}$, $n \geq 2$, the integrable Hamiltonian $\sum_{j\in\mathbb{Z}} \lambda_j u_j \bar u_j$ has the invariant tori $\{u_j \bar u_j = \xi_j$, for $j \in \mathcal{I}$, $u_j = \bar u_j = 0$ for $j \notin \mathcal{I}\}$ parametrized by the actions $\xi = (\xi_j)_{j\in\mathcal{I}} \in \mathbb{R}^n$. The next KAM result states the existence of nearby invariant tori for the complete Hamiltonian H in (1.4).

THEOREM 1.1. – *The Equation* (1.1)-(1.2) *admits Cantor families of small-amplitude, analytic, quasi-periodic solutions with zero Lyapunov exponents and whose linearized equation is reducible to constant coefficients. Such Cantor families have asymptotically full measure at the origin in the set of parameters.*

The proof of Theorem 1.1 is based on the abstract infinite dimensional KAM Theorem 4.1, which provides a reducible normal form (see (4.12)) around the elliptic invariant torus, and on the measure estimates Theorem 4.2. The key point in proving Theorem 4.2 is the asymptotic bound (4.9) on the perturbed normal frequencies $\Omega^\infty(\xi)$ after the KAM iteration. This allows to prove that the second order Melnikov non-resonance conditions (4.11) are fulfilled for an asymptotically full measure set of parameters (see (4.16)). The estimate (4.9), in turn, is achieved by exploiting the *quasi-Töplitz* property of the perturbation. This notion has been introduced by Procesi-Xu [30] in the context of NLS in higher space dimensions and it is similar, in spirit, to the Töplitz-Lipschitz property in Eliasson-Kuksin [14]. The precise formulation of quasi-Töplitz functions, adapted to the DNLW setting, is given in Definition 3.4 below.

Let us roughly explain the main ideas and techniques for proving Theorems 4.1, 4.2. These theorems concern, as usual, a parameter dependent family of analytic Hamiltonians of the form

$$H = \omega(\xi) \cdot y + \Omega(\xi) \cdot z\bar{z} + P(x, y, z, \bar{z}; \xi) \tag{1.6}$$

where $(x, y) \in \mathbb{T}^n \times \mathbb{R}^n$, $z, \bar{z}$ are infinitely many variables, $\omega(\xi) \in \mathbb{R}^n$, $\Omega(\xi) \in \mathbb{R}^\infty$ and $\xi \in \mathbb{R}^n$. The frequencies $\Omega_j(\xi)$ are close to the unperturbed frequencies λ_j in (1.5).

As is well known, the main difficulty of the KAM iteration which provides a reducible KAM normal form like (4.12) is to fulfill, at each iterative step, the second order Melnikov non-resonance conditions. Actually, following the formulation of the KAM theorem given in [2], it is sufficient to verify

$$|\omega^\infty(\xi) \cdot k + \Omega_i^\infty(\xi) - \Omega_j^\infty(\xi)| \geq \frac{\gamma}{1 + |k|^\tau}, \quad \gamma > 0, \tag{1.7}$$

only for the "final" frequencies $\omega^\infty(\xi)$ and $\Omega^\infty(\xi)$, see (4.11), and not along the inductive iteration.

The application of the usual KAM theory (see e.g., [20], [27]-[28]), to the DNLW equation provides only the asymptotic decay estimate

$$\Omega_j^\infty(\xi) = j + O(1) \quad \text{for} \quad j \to +\infty. \tag{1.8}$$

Such a bound is not enough: the set of parameters ξ satisfying (1.7) could be empty. Note that for the semilinear NLW equation (see e.g., [27]) the frequencies decay asymptotically faster, namely like $\Omega_j^\infty(\xi) = j + O(1/j)$.

The key idea for verifying the second order Melnikov non-resonance conditions (1.7) for DNLW is to prove the higher order asymptotic decay estimate (see (4.9), (4.2))

$$\Omega_j^\infty(\xi) = j + a_+(\xi) + \frac{\mathrm{m}}{2j} + O(\frac{\gamma^{2/3}}{j}) \quad \text{for} \quad j \geq O(\gamma^{-1/3}) \tag{1.9}$$

where $a_+(\xi)$ is a constant independent of j (an analogous expansion holds for $j \to -\infty$ with a possibly different limit constant $a_-(\xi)$). In this way infinitely many conditions in (1.7) are verified by imposing only first order Melnikov conditions like $|\omega^\infty(\xi) \cdot k + h| \geq 2\gamma^{2/3}/|k|^\tau$, $h \in \mathbb{Z}$. Indeed, for $i > j > O(|k|^\tau \gamma^{-1/3})$, we get

$$\begin{aligned} |\omega^\infty(\xi) \cdot k + \Omega_i^\infty(\xi) - \Omega_j^\infty(\xi)| &= |\omega^\infty(\xi) \cdot k + i - j + \frac{\mathrm{m}(i-j)}{2ij} + O(\gamma^{2/3}/j)| \\ &\geq 2\gamma^{2/3}|k|^{-\tau} - O(|k|/j^2) - O(\gamma^{2/3}/j) \geq \gamma^{2/3}|k|^{-\tau} \end{aligned}$$

noting that $i - j$ is integer and $|i - j| = O(|k|)$ (otherwise no small divisors occur). We refer to Section 6 for the precise arguments, see in particular Lemma 6.2.

The asymptotic decay (4.9) for the perturbed frequencies $\Omega^\infty(\xi)$ is achieved thanks to the "quasi-Töplitz" property of the perturbation (Definition 3.4). Let us roughly explain this