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ARITHMETIC OF 0-CYCLES ON VARIETIES DEFINED OVER NUMBER FIELDS

BY YONGQI LIANG

ABSTRACT. – Let X be a rationally connected algebraic variety, defined over a number field k. We find a relation between the arithmetic of rational points on X and the arithmetic of zero-cycles. More precisely, we consider the following statements: (1) the Brauer-Manin obstruction is the only obstruction to weak approximation for K-rational points on X_K for all finite extensions K/k; (2) the Brauer-Manin obstruction is the only obstruction to weak approximation in some sense that we define for zero-cycles of degree 1 on X_K for all finite extensions K/k; (3) the sequence

$$\varprojlim_n CH_0(X_K)/n \to \prod_{w\in\Omega_K} \varprojlim_n CH'_0(X_{K_w})/n \to \mathrm{Hom}(\mathrm{Br}(X_K), \mathbb{Q}/\mathbb{Z})$$

is exact for all finite extensions K/k. We prove that (1) implies (2), and that (2) and (3) are equivalent. We also prove a similar implication for the Hasse principle.

As an application, we prove the exactness of the sequence above for smooth compactifications of certain homogeneous spaces of linear algebraic groups.

RÉSUMÉ. – Soit X une variété algébrique rationnellement connexe, définie sur un corps de nombres k. On trouve, sur X, un lien entre l'arithmétique des points rationnels et l'arithmétique des zéro-cycles. Plus précisément, on considère les assertions suivantes: (1) l'obstruction de Brauer-Manin est la seule à l'approximation faible pour les points K-rationnels sur X_K pour toute extension finie K/k; (2) l'obstruction de Brauer-Manin est la seule à l'approximation faible (en un certain sens à préciser) pour les zéro-cycles de degré 1 sur X_K pour toute extension finie K/k; (3) la suite

$$\varprojlim_n CH_0(X_K)/n \to \prod_{w\in\Omega_K} \varprojlim_n CH'_0(X_{K_w})/n \to \mathrm{Hom}(\mathrm{Br}(X_K), \mathbb{Q}/\mathbb{Z})$$

est exacte pour toute extension finie K/k. On démontre que (1) implique (2), et que (2) et (3) sont équivalentes. On trouve également une implication similaire pour le principe de Hasse.

Comme application, on montre l'exactitude de la suite ci-dessus pour les compactifications lisses de certains espaces homogènes de groupes algébriques linéaires.

Introduction

Brauer-Manin obstruction, for rational points and for 0-cycles

Let X be a proper smooth algebraic variety defined over a number field k. We denote by Ω_k the set of all the places of k, and by k_v the completion of k with respect to $v \in \Omega_k$. Let $\mathrm{Br}(X) = H^2_{\text{ét}}(X, \mathbb{G}_m)$ be the cohomological Brauer group of X. In [33], Manin defined the following pairing (called the *Brauer-Manin pairing*):

$$\begin{array}{rcl} \langle\cdot,\cdot\rangle_k : \prod_{v\in\Omega_k} X(k_v) \times \mathrm{Br}(X) & \to & \mathbb{Q}/\mathbb{Z}, \\ (\ \{x_v\}_{v\in\Omega_k} \quad , \quad b\ ) & \mapsto & \sum_{v\in\Omega_k} \mathrm{inv}_v(b(x_v)), \end{array}$$

where $\mathrm{inv}_v : \mathrm{Br}(k_v) \hookrightarrow \mathbb{Q}/\mathbb{Z}$ is the local invariant at v, and where $b(x_v)$ is the evaluation of b at the point x_v, *i.e.* the pull-back of $b \in \mathrm{Br}(X)$ via the morphism $x_v : \mathrm{Spec}(k_v) \to X$. According to the exact sequence

$$0 \to \mathrm{Br}(k) \to \bigoplus_v \mathrm{Br}(k_v) \to \mathbb{Q}/\mathbb{Z} \to 0$$

coming from class field theory, the pairing factorizes through $\mathrm{Br}(X)/\mathrm{Br}(k)$; and the left kernel $[\prod_v X(k_v)]^{\mathrm{Br}}$ of the pairing contains $X(k)$, and its closure $\overline{X(k)}$ in $\prod_v X(k_v)$ by continuity. The condition $[\prod_v X(k_v)]^{\mathrm{Br}} = \varnothing$ gives an obstruction to the existence of a global rational point. This explained most known examples for the failure of Hasse principle at that time. If $[\prod_v X(k_v)]^{\mathrm{Br}} \neq \varnothing$ implies that X has a k-rational point, we say that *the Brauer-Manin obstruction is the only obstruction to the Hasse principle for rational points*. Similarly, the condition $[\prod_v X(k_v)]^{\mathrm{Br}} \subsetneq \prod_v X(k_v)$ gives an obstruction to the weak approximation property; if $\overline{X(k)} = [\prod_v X(k_v)]^{\mathrm{Br}}$ we say that *the Brauer-Manin obstruction is the only obstruction to weak approximation for rational points*.

Question. – *On which families of algebraic varieties, is the Brauer-Manin obstruction the only obstruction to the Hasse principle/weak approximation for rational points?*

The question dates back to a paper of Colliot-Thélène and Sansuc [10], see the Bourbaki exposé by Peyre [34] for a recent survey of the problem.

Let us write $X_K = X \otimes_k K$ for any field extension K of k. In [4], Colliot-Thélène extended the Brauer-Manin pairing to 0-cycles:

$$\begin{array}{rcl} \langle\cdot,\cdot\rangle_k : \prod_{v\in\Omega_k} Z_0(X_{k_v}) \times \mathrm{Br}(X) & \to & \mathbb{Q}/\mathbb{Z}, \\ (\ \{z_v\}_{v\in\Omega_k} \quad , \quad b\ ) & \mapsto & \sum_{v\in\Omega_k} \mathrm{inv}_v(\langle z_v, b\rangle_{k_v}), \end{array}$$

where $\langle\cdot,\cdot\rangle_{k_v}$ is defined as follows,

$$\begin{array}{rcl} \langle\cdot,\cdot\rangle_{k_v} : Z_0(X_{k_v}) \times \mathrm{Br}(X) & \to & \mathrm{Br}(k_v), \\ (\ \sum_P n_P P \quad , \quad b\ ) & \mapsto & \sum_P n_P \mathrm{cores}_{k_v(P)/k_v}(b(P)). \end{array}$$

He conjectured that the Brauer-Manin obstruction is the only obstruction to the Hasse principle for 0-cycles of degree 1 on any smooth proper variety. This conjecture is proved for any curve (supposing finiteness of the Tate-Shafarevich group of its Jacobian) by Saito [36].

As a variant of [14, page 69], we say that *the Brauer-Manin obstruction is the only obstruction to weak approximation for 0-cycles of degree* 1, if for any positive integer n and for any finite subset $S \subset \Omega_k$, given an arbitrary family $\{z_v\} \perp \mathrm{Br}(X)$ of local 0-cycles of degree 1, then there exists a global 0-cycle $z_{n,S}$ of degree 1 such that $z_{n,S}$ and z_v have the same image in $CH_0(X_{k_v})/n$ for any $v \in S$, *cf.* §1.2. This notion turns out to be closely related to the exactness of the sequences (E) and (E_0) described as follows.

Exact sequences of local-global type

We define the modified Chow group $CH_0'(X_{k_v})$ to be the usual Chow group $CH_0(X_{k_v})$ of 0-cycles for each non-archimedean place v, otherwise we set

$$CH_0'(X_{k_v}) = CH_0(X_{k_v})/N_{\bar{k}_v|k_v} CH_0(X_{\bar{k}_v})$$

for archimedean places where $N_{\bar{k}_v|k_v}$ represents the norm. The above pairing for 0-cycles factorizes through the modified Chow groups $CH_0'(X_{k_v})$. Let us denote $M^\wedge = \varprojlim_n M/nM$ for any abelian group M. The exactness of the complex induced by the Brauer-Manin pairing

$$(E) \qquad CH_0(X)^\wedge \to \prod_{v\in\Omega} CH_0'(X_{k_v})^\wedge \to \mathrm{Hom}(\mathrm{Br}(X), \mathbb{Q}/\mathbb{Z})$$

is conjectured by Colliot-Thélène/Sansuc [11] for rational surfaces, by Kato/Saito [26] and Colliot-Thélène [4] for all smooth proper varieties; it has been reformulated into this form by van Hamel [21] and Wittenberg [43].

Let $A_0(X) = \mathrm{Ker}[\deg : CH_0(X) \to \mathbb{Z}]$. We can consider the degree 0 part of the sequence (E):

$$(E_0). \qquad A_0(X)^\wedge \to \prod_{v\in\Omega} A_0(X_{k_v})^\wedge \to \mathrm{Hom}(\mathrm{Br}(X), \mathbb{Q}/\mathbb{Z})$$

If X is a smooth projective curve such that the Tate-Shafarevich group $Ш(\mathrm{Jac}(X), k)$ of its Jacobian is finite, then Colliot-Thélène proved the exactness of (E) for X in [5, Proposition 3.3]. In particular, he reproved (part of) the Cassels-Tate exact sequence when X is an elliptic curve.

Rational points versus 0-cycles

One may ask whether there are some relations between the arithmetic of rational points and the arithmetic of 0-cycles. Some recent examples (*cf.* §4) show that one should not expect an affirmative answer for general proper smooth varieties. But sometimes these two questions are discussed in parallel using similar methods, for example [14] [13].

Main results

In the present work, mainly for rationally connected varieties, we find out a relation between the Brauer-Manin obstructions for rational points and for 0-cycles on rationally connected varieties. As an application, we prove the exactness of (E) and (E_0) for certain homogeneous spaces of linear algebraic groups. To be precise, we state our main results as follows.

Let X be a proper smooth variety defined over a number field k. Consider the following statements:

- (pt-HP) *the Brauer-Manin obstruction is the only obstruction to the Hasse principle for* K*-rational points on* X_K *for any finite extension* K/k;
- (pt-WA) *the Brauer-Manin obstruction is the only obstruction to weak approximation for* K*-rational points on* X_K *for any finite extension* K/k;
- (0cyc-HP) *the Brauer-Manin obstruction is the only obstruction to the Hasse principle for 0-cycles of degree* 1 *on* X_K *for any finite extension* K/k;
- (0cyc-WA) *the Brauer-Manin obstruction is the only obstruction to weak approximation for 0-cycles of degree* 1 *on* X_K *for any finite extension* K/k;
- (Exact) *the sequence* (E) *is exact for* X_K, *and therefore so is* (E_0), *for any finite extension* K/k.

Theorem A. – *The statement* (Exact) *implies* (0cyc-HP) *and* (0cyc-WA). *If moreover* X *is* rationally connected, *then* (0cyc-WA) *is equivalent to* (Exact).

Theorem B (Theorem 3.2.1). – *If* X *is a variety such that the Néron-Severi group* $NS(X_{\bar{k}})$ *is torsion-free and* $H^1(X_{\bar{k}}, \mathcal{O}_{X_{\bar{k}}}) = H^2(X_{\bar{k}}, \mathcal{O}_{X_{\bar{k}}}) = 0$, *in particular if* X *is* rationally connected, *then* (pt-HP) *implies* (0cyc-HP), *and* (pt-WA) *implies* (0cyc-WA).

Since the statements (pt-HP) and (pt-WA) are proved by Borovoi for smooth compactifications of certain homogeneous spaces of linear algebraic groups [1], we obtain the following

Corollary. – *Let* Y *be a homogeneous space of a connected linear algebraic group* G *defined over a number field, and let* X *be a smooth compactification of* Y. *Suppose that the geometric stabilizer of* Y *is connected (or abelian if* G *is supposed semisimple simply connected).*

Then the sequences (E) *and* (E_0) *are exact for* X, *and the Brauer-Manin obstruction is the only obstruction to the Hasse principle/weak approximation for 0-cycles of degree* 1 *on* X.

The exactness of (E) was not known before the present article even for (smooth compactifications of) algebraic tori of arbitrary dimension.

Organization of the article

After some preliminaries in §1, we discuss in §2 the relation between the exactness of the sequence (E) and weak approximation for 0-cycles, and prove the first assertion of Theorem A. In §3 assuming rational connectedness we give a detailed proof of the second assertion of Theorem A as well as Theorem B, which divides into several steps with more precise statements. The proof of Corollary is given at the end of §3. Finally in §4, we make some remarks on Theorems A, B, and the corollary.

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