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LAGRANGIAN FIBRATIONS ON HYPERKÄHLER MANIFOLDS – ON A QUESTION OF BEAUVILLE

BY DANIEL GREB, CHRISTIAN LEHN AND SÖNKE ROLLENSKE

ABSTRACT. – Let X be a compact hyperkähler manifold containing a complex torus L as a Lagrangian subvariety. Beauville posed the question whether X admits a Lagrangian fibration with fibre L. We show that this is indeed the case if X is not projective. If X is projective we find an almost holomorphic Lagrangian fibration with fibre L under additional assumptions on the pair (X, L), which can be formulated in topological or deformation-theoretic terms. Moreover, we show that for any such almost holomorphic Lagrangian fibration there exists a smooth good minimal model, i.e., a hyperkähler manifold birational to X on which the fibration is holomorphic.

RÉSUMÉ. – Soit X une variété hyperkählérienne compacte contenant un tore complexe L en tant que sous-variété lagrangienne. A. Beauville a posé la question suivante : la variété X admet-elle une fibration lagrangienne de fibre L? Nous démontrons que c'est le cas si X n'est pas projective. Si X est projective nous montrons l'existence d'une fibration lagrangienne presque holomorphe de fibre L sous des hypothèses plus restrictives sur la paire (X, L). Ces hypothèses peuvent se formuler de deux manières : en termes topologiques ou grâce à la théorie des déformations de (X, L). Par ailleurs, nous démontrons que pour une telle fibration lagrangienne presque holomorphe il y a toujours un bon modèle minimal lisse, c'est-à-dire une variété hyperkählérienne birationelle à X sur laquelle la fibration est holomorphe.

Introduction

By the classical decomposition theorem of Beauville-Bogomolov, every compact Kähler manifold with vanishing first Chern class admits a finite cover which decomposes as a product of tori, Calabi-Yau manifolds, and hyperkähler manifolds, see e.g., [5, Thm. 1]. While tori are quite well-understood, a classification of Calabi-Yau and hyperkähler manifolds is still far out of reach. Only in dimension 2, where Calabi-Yau and hyperkähler manifolds coincide, the theory of K3-surfaces provides a fairly complete picture.

Let now X be a hyperkähler manifold, that is, a compact, simply-connected Kähler manifold X such that $H^0(X, \Omega_X^2)$ is spanned by a holomorphic symplectic form σ. From a differential geometric point of view hyperkähler manifolds are Riemannian manifolds with holonomy the full unitary-symplectic group $\mathrm{Sp}(n)$.

An important step in the structural understanding of a manifold is to decide whether there is a fibration $f\colon X \to B$ over a complex space of smaller dimension. For hyperkähler manifolds it is known that in case such f exists, it is a *Lagrangian fibration*: $\dim X = 2\dim B$, and the holomorphic symplectic form σ restricts to zero on the general fibre. Additionally, by the Arnold-Liouville theorem the general fibre is a smooth Lagrangian torus, see Section 1.2 for a detailed discussion.

In accordance with the case of K3-surfaces (and also motivated by mirror symmetry) a simple version of the so-called Hyperkähler SYZ-conjecture[(1)] asks if every hyperkähler manifold can be deformed to a hyperkähler manifold admitting a Lagrangian fibration. With this as a starting point, an approach to a rough classification of hyperkähler manifolds has been proposed, see e.g., [41]. A more sophisticated version of the SYZ-conjecture is discussed in Section 6.1.

Here we approach the question of existence of a Lagrangian fibration on a given hyperkähler manifold X under a geometric assumption proposed by Beauville [7, Sect. 1.6]:

QUESTION B. – *Let X be a hyperkähler manifold and $L \subset X$ a Lagrangian submanifold biholomorphic to a complex torus. Is L a fibre of a (meromorphic) Lagrangian fibration $f\colon X \to B$?*

Building on work of Campana, Oguiso, and Peternell [10] we give a positive answer in case X is not projective.

THEOREM 4.1. – *Let X be a non-projective hyperkähler manifold of dimension $2n$ containing a Lagrangian subtorus L. Then the algebraic dimension of X is n, and there exists an algebraic reduction $f\colon X \to B$ of X that is a holomorphic Lagrangian fibration with fibre L.*

In the case of *projective* hyperkähler manifold X containing a Lagrangian subtorus L, we work out a necessary and sufficient criterion for the existence of an almost holomorphic fibration with fibre L, i.e., for a slightly weaker positive answer to Beauville's question.

THEOREM 5.3. – *Let X be a projective hyperkähler manifold and $L \subset X$ a Lagrangian subtorus. Then the following are equivalent.*

1. *X admits an almost holomorphic Lagrangian fibration with strong fibre L.*
2. *The pair (X, L) admits a small deformation (X', L') with non-projective X'.*
3. *There exists an effective divisor D on X such that $c_1(\mathscr{O}_X(D)|_L) = 0 \in H^{1,1}(L, \mathbb{R})$.*

Here, *strong fibre* means that f is holomorphic near L, and L is a fibre of the corresponding holomorphic map. The proof of Theorem 5.3 consists of two major steps: First, assuming the existence of a small deformation of (X, L) to a non-projective pair (X', L'), we use Theorem 4.1 to produce a Lagrangian fibration with fibre L' on X' and then degenerate this fibration to an almost-holomorphic fibration on (X, L) using relative Barlet spaces. Second, the existence of a small deformation to a non-projective pair (X', L') is characterised in terms

(1) We refer the reader to [42] for a historical discussion concerning the emergence of this conjecture.

of periods in $H^2(X, \mathbb{C})$. This finally leads to the condition on the existence of a special divisor, as stated in part (iii) of Theorem 5.3.(2)

From the discussion above the question arises how far an almost holomorphic fibration is away from answering Beauville's question in the strong form. If $f : X \dashrightarrow B$ is an almost holomorphic Lagrangian fibration, then it is natural to search for a holomorphic model of f in the same birational equivalence class. This is done in the final section, where using the recent advances in higher-dimensional birational geometry ([8, 19]) the following result is proven.

Theorem (see Theorem 6.3). – *Let X be a projective hyperkähler manifold with an almost holomorphic Lagrangian fibration $f : X \dashrightarrow B$. Then there exists a holomorphic model for f on a birational hyperkähler manifold X'. In other words, there is a commutative diagram*

$$\begin{tikzcd} X \arrow[r, dashed] \arrow[d, dashed, "f"'] & X' \arrow[d, "f'"] \\ B \arrow[r, dashed] & B' \end{tikzcd}$$

where f' is a holomorphic Lagrangian fibration on X' and the horizontal maps are birational.

Theorem 6.3 proves a special version of the Hyperkähler SYZ-conjecture. Related results were obtained by Amerik and Campana [1, Thm. 3.6] in dimension four. Note furthermore that birational hyperkähler manifolds are deformation-equivalent by work of Huybrechts [20, Thm. 4.6], so Theorem 6.3 might also lead to a new approach to the general case of the Hyperkähler SYZ-conjecture.

The connection to this circle of ideas is also manifest in the following generalization of a result of Matsushita, which we obtain as a corollary of Theorem 6.3.

Theorem 6.12. – *Let X be a projective hyperkähler manifold and $f : X \dashrightarrow B$ an almost holomorphic map with connected fibres onto a normal projective variety B. If $0 < \dim B < \dim X$, then $\dim B = \frac{1}{2}\dim X$, and f is an almost holomorphic Lagrangian fibration.*

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(2) After this article was written, Jun-Muk Hwang and Richard Weiss posted a preprint [23] in which they prove that the criterion given in part (iii) of Theorem 5.3 is fulfilled for any projective hyperkähler manifold X containing a Lagrangian subtorus L. See also Remark 6.14.

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1. Preliminaries on hyperkähler manifolds

We collect a few basic definitions and properties of the objects of our study.

DEFINITION 1.1. – An *irreducible holomorphic symplectic manifold* or *hyperkähler manifold* is a simply-connected compact Kähler manifold X such that $H^0(X, \Omega_X^2)$ is spanned by an everywhere non-degenerate holomorphic two-form σ.

Actually, the notion of hyperkähler manifold is of differential-geometric origin and stands for a Ricci-flat Kähler manifold with holonomy group $\mathrm{Sp}(n)$. It was shown by Beauville in [5, Prop 4] that this condition is equivalent to the existence of a holomorphic symplectic form unique up to scalars; often the terms *irreducible holomorphic symplectic manifold* and *hyperkähler manifold* are therefore used synonymously.

1.1. The Beauville-Bogomolov form

The second cohomology $H^2(X, \mathbb{Z})$ of a hyperkähler manifold X carries a natural, integral symmetric bilinear form

$$q = q_X : H^2(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) \to \mathbb{Z},$$

the so-called *Beauville-Bogomolov-Fujiki form* (see [5, Thm. 5] or [21, Def. 22.8]). Since we need to consider the restriction of this form to subspaces where it might be degenerate, we give its signature as a triple containing (in this order) the number of positive, zero, and negative eigenvalues of the associated real symmetric bilinear form. In this notation q has signature $(3, 0, b_2(X) - 3)$, and its restriction to $H^{1,1}(X, \mathbb{R})$ has signature $(1, 0, h^{1,1} - 1)$, see [21, Cor. 23.11].

Let $\rho = \rho(X)$ be the Picard number of X, that is, the rank of the Néron-Severi group $\mathrm{NS}(X) = H^{1,1}(X) \cap H^2(X, \mathbb{Q})$. We distinguish hyperkähler manifolds according to the signature of the restriction of q to $\mathrm{NS}(X)$. We call X *hyperbolic* if $q|_{\mathrm{NS}(X)}$ has signature $(1, 0, \rho - 1)$, *parabolic* if $q|_{\mathrm{NS}(X)}$ has signature $(0, 1, \rho - 1)$, and *elliptic* if $q|_{\mathrm{NS}(X)}$ has signature $(0, 0, \rho)$. The relevance of these notions is underlined by the following result of Huybrechts.

THEOREM 1.2 (Prop. 26.13 of [21]). – *A hyperkähler manifold X is projective if and only if X is hyperbolic.*