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Lagrangian fibrations on hyperkähler manifolds

On a question of Beauville

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
LAGRANGIAN FIBRATIONS
ON HYPERKÄHLER MANIFOLDS –
ON A QUESTION OF BEAUVILLE

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ABSTRACT. – Let $X$ be a compact hyperkähler manifold containing a complex torus $L$ as a Lagrangian subvariety. Beauville posed the question whether $X$ admits a Lagrangian fibration with fibre $L$. We show that this is indeed the case if $X$ is not projective. If $X$ is projective we find an almost holomorphic Lagrangian fibration with fibre $L$ under additional assumptions on the pair $(X, L)$, which can be formulated in topological or deformation-theoretic terms. Moreover, we show that for any such almost holomorphic Lagrangian fibration there exists a smooth good minimal model, i.e., a hyperkähler manifold birational to $X$ on which the fibration is holomorphic.

RéSUMÉ. – Soit $X$ une variété hyperkählérienne compacte contenant un tore complexe $L$ en tant que sous-variété lagrangienne. A. Beauville a posé la question suivante : la variété $X$ admet-elle une fibration lagrangienne de fibre $L$? Nous démontrons que c’est le cas si $X$ n’est pas projective. Si $X$ est projective nous montrons l’existence d’une fibration lagrangienne presque holomorphe de fibre $L$ sous des hypothèses plus restrictives sur la paire $(X, L)$. Ces hypothèses peuvent se formuler de deux manières : en termes topologiques ou grâce à la théorie des déformations de $(X, L)$. Par ailleurs, nous démontrons que pour une telle fibration lagrangienne presque holomorphe il y a toujours un bon modèle minimal lisse, c’est-à-dire une variété hyperkählérienne birationnelle à $X$ sur laquelle la fibration est holomorphe.

Introduction

By the classical decomposition theorem of Beauville-Bogomolov, every compact Kähler manifold with vanishing first Chern class admits a finite cover which decomposes as a product of tori, Calabi-Yau manifolds, and hyperkähler manifolds, see e.g., [5, Thm. 1]. While tori are quite well-understood, a classification of Calabi-Yau and hyperkähler manifolds is still far out of reach. Only in dimension 2, where Calabi-Yau and hyperkähler manifolds coincide, the theory of K3-surfaces provides a fairly complete picture.

Let now $X$ be a hyperkähler manifold, that is, a compact, simply-connected Kähler manifold $X$ such that $H^0(X, \Omega_X^2)$ is spanned by a holomorphic symplectic form $\sigma$. From a
differential geometric point of view hyperkähler manifolds are Riemannian manifolds with holonomy the full unitary-symplectic group \( \text{Sp}(n) \).

An important step in the structural understanding of a manifold is to decide whether there is a fibration \( f : X \to B \) over a complex space of smaller dimension. For hyperkähler manifolds it is known that in case such \( f \) exists, it is a Lagrangian fibration: \( \dim X = 2 \dim B \), and the holomorphic symplectic form \( \sigma \) restricts to zero on the general fibre. Additionally, by the Arnold-Liouville theorem the general fibre is a smooth Lagrangian torus, see Section 1.2 for a detailed discussion.

In accordance with the case of K3-surfaces (and also motivated by mirror symmetry) a simple version of the so-called Hyperkähler SYZ-conjecture\(^{(1)}\) asks if every hyperkähler manifold can be deformed to a hyperkähler manifold admitting a Lagrangian fibration. With this as a starting point, an approach to a rough classification of hyperkähler manifolds has been proposed, see e.g., [41]. A more sophisticated version of the SYZ-conjecture is discussed in Section 6.1.

Here we approach the question of existence of a Lagrangian fibration on a given hyperkähler manifold \( X \) under a geometric assumption proposed by Beauville [7, Sect. 1.6]:

**Question B.** – Let \( X \) be a hyperkähler manifold and \( L \subset X \) a Lagrangian submanifold biholomorphic to a complex torus. Is \( L \) a fibre of a (meromorphic) Lagrangian fibration \( f : X \to B \)?

Building on work of Campana, Oguiso, and Peternell [10] we give a positive answer in case \( X \) is not projective.

**Theorem 4.1.** – Let \( X \) be a non-projective hyperkähler manifold of dimension \( 2n \) containing a Lagrangian subtorus \( L \). Then the algebraic dimension of \( X \) is \( n \), and there exists an algebraic reduction \( f : X \to B \) of \( X \) that is a holomorphic Lagrangian fibration with fibre \( L \).

In the case of projective hyperkähler manifold \( X \) containing a Lagrangian subtorus \( L \), we work out a necessary and sufficient criterion for the existence of an almost holomorphic fibration with fibre \( L \), i.e., for a slightly weaker positive answer to Beauville’s question.

**Theorem 5.3.** – Let \( X \) be a projective hyperkähler manifold and \( L \subset X \) a Lagrangian subtorus. Then the following are equivalent.

1. \( X \) admits an almost holomorphic Lagrangian fibration with strong fibre \( L \).
2. The pair \( (X, L) \) admits a small deformation \( (X', L') \) with non-projective \( X' \).
3. There exists an effective divisor \( D \) on \( X \) such that \( c_1(\mathcal{O}_X(D)|_L) = 0 \in H^{1,1}(L, \mathbb{R}) \).

Here, strong fibre means that \( f \) is holomorphic near \( L \), and \( L \) is a fibre of the corresponding holomorphic map. The proof of Theorem 5.3 consists of two major steps: First, assuming the existence of a small deformation of \( (X, L) \) to a non-projective pair \( (X', L') \), we use Theorem 4.1 to produce a Lagrangian fibration with fibre \( L' \) on \( X' \) and then degenerate this fibration to an almost-holomorphic fibration on \( (X, L) \) using relative Barlet spaces. Second, the existence of a small deformation to a non-projective pair \( (X', L') \) is characterised in terms

\( ^{(1)} \) We refer the reader to [42] for a historical discussion concerning the emergence of this conjecture.
of periods in $H^2(X, \mathbb{C})$. This finally leads to the condition on the existence of a special divisor, as stated in part (iii) of Theorem 5.3.\(^{(2)}\)

From the discussion above the question arises how far an almost holomorphic fibration is away from answering Beauville’s question in the strong form. If $f : X \rightarrow B$ is an almost holomorphic Lagrangian fibration, then it is natural to search for a holomorphic model of $f$ in the same birational equivalence class. This is done in the final section, where using the recent advances in higher-dimensional birational geometry ([8, 19]) the following result is proven.

**Theorem** (see Theorem 6.3). – *Let $X$ be a projective hyperkähler manifold with an almost holomorphic Lagrangian fibration $f : X \rightarrow B$. Then there exists a holomorphic model for $f$ on a birational hyperkähler manifold $X'$. In other words, there is a commutative diagram*

$$
\begin{array}{ccc}
X & \rightarrow & X' \\
\downarrow f & & \downarrow f' \\
B & \rightarrow & B'
\end{array}
$$

*where $f'$ is a holomorphic Lagrangian fibration on $X'$ and the horizontal maps are birational.*

Theorem 6.3 proves a special version of the Hyperkähler SYZ-conjecture. Related results were obtained by Amerik and Campana [1, Thm. 3.6] in dimension four. Note furthermore that birational hyperkähler manifolds are deformation-equivalent by work of Huybrechts [20, Thm. 4.6], so Theorem 6.3 might also lead to a new approach to the general case of the Hyperkähler SYZ-conjecture.

The connection to this circle of ideas is also manifest in the following generalization of a result of Matsushita, which we obtain as a corollary of Theorem 6.3.

**Theorem 6.12.** – *Let $X$ be a projective hyperkähler manifold and $f : X \rightarrow B$ an almost holomorphic map with connected fibres onto a normal projective variety $B$. If $0 < \dim B < \dim X$, then $\dim B = \frac{1}{2} \dim X$, and $f$ is an almost holomorphic Lagrangian fibration.*

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\(^{(2)}\) After this article was written, Jun-Muk Hwang and Richard Weiss posted a preprint [23] in which they prove that the criterion given in part (iii) of Theorem 5.3 is fulfilled for any projective hyperkähler manifold $X$ containing a Lagrangian subtorus $L$. See also Remark 6.14.
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1. Preliminaries on hyperkähler manifolds

We collect a few basic definitions and properties of the objects of our study.

**Definition 1.1.** – An irreducible holomorphic symplectic manifold or hyperkähler manifold is a simply-connected compact Kähler manifold $X$ such that $H^0(X, \Omega^2_X)$ is spanned by an everywhere non-degenerate holomorphic two-form $\sigma$.

Actually, the notion of hyperkähler manifold is of differential-geometric origin and stands for a Ricci-flat Kähler manifold with holonomy group $\text{Sp}(n)$. It was shown by Beauville in [5, Prop. 4] that this condition is equivalent to the existence of a holomorphic symplectic form unique up to scalars; often the terms irreducible holomorphic symplectic manifold and hyperkähler manifold are therefore used synonymously.

1.1. The Beauville-Bogomolov form

The second cohomology $H^2(X, \mathbb{Z})$ of a hyperkähler manifold $X$ carries a natural, integral symmetric bilinear form

$$q = q_X : H^2(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) \to \mathbb{Z},$$

the so-called **Beauville-Bogomolov-Fujiki form** (see [5, Thm. 5] or [21, Def. 22.8]). Since we need to consider the restriction of this form to subspaces where it might be degenerate, we give its signature as a triple containing (in this order) the number of positive, zero, and negative eigenvalues of the associated real symmetric bilinear form. In this notation $q$ has signature $(3, 0, b_2(X) - 3)$, and its restriction to $H^{1,1}(X, \mathbb{R})$ has signature $(1, 0, h^{1,1} - 1)$, see [21, Cor. 23.11].

Let $\rho = \rho(X)$ be the Picard number of $X$, that is, the rank of the Néron-Severi group $\text{NS}(X) = H^{1,1}(X) \cap H^2(X, \mathbb{Q})$. We distinguish hyperkähler manifolds according to the signature of the restriction of $q$ to $\text{NS}(X)$. We call $X$ hyperbolic if $q|_{\text{NS}(X)}$ has signature $(1, 0, \rho - 1)$, parabolic if $q|_{\text{NS}(X)}$ has signature $(0, 1, \rho - 1)$, and elliptic if $q|_{\text{NS}(X)}$ has signature $(0, 0, \rho)$. The relevance of these notions is underlined by the following result of Huybrechts.

**Theorem 1.2 (Prop. 26.13 of [21]).** – A hyperkähler manifold $X$ is projective if and only if $X$ is hyperbolic.