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Baptiste CALMÈS & Viktor PETROV & Kirill ZAINOULLINE

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INVARIANTS, TORSION INDICES AND ORIENTED COHOMOLOGY OF COMPLETE FLAGS

BY Baptiste CALMÈS, Viktor PETROV
AND Kirill ZAINOULLINE

ABSTRACT. – Let G be a split semisimple linear algebraic group over a field and let T be a split maximal torus of G. Let h be an oriented cohomology (algebraic cobordism, connective K-theory, Chow groups, Grothendieck's K_0, etc.) with formal group law F. We construct a ring from F and the characters of T, that we call a formal group ring, and we define a characteristic ring morphism c from this formal group ring to $\mathsf{h}(G/B)$ where G/B is the variety of Borel subgroups of G. Our main result says that when the torsion index of G is inverted, c is surjective and its kernel is generated by elements invariant under the Weyl group of G. As an application, we provide an algorithm to compute the ring structure of $\mathsf{h}(G/B)$ and to describe the classes of desingularized Schubert varieties and their products.

RÉSUMÉ. – Soit G un groupe algébrique linéaire semi-simple déployé sur un corps et soit T un tore maximal déployé de G. Étant donnée une cohomologie orientée h (anneau de Chow, K_0 de Grothendieck, K-théorie connective, etc.) et sa loi de groupe formel F, nous construisons un anneau appelé anneau de groupe formel, associé à F et aux caractères de T, puis un homomorphisme caractéristique c de cet anneau de groupe formel vers l'anneau $\mathsf{h}(G/B)$ où G/B est la variété des sous-groupes de Borel de G. Le résultat principal de cet article montre que, lorsque l'indice de torsion du groupe G est inversé, c est surjectif et son noyau est engendré par des éléments invariants sous l'action du groupe de Weyl de G. En guise d'application, nous fournissons un algorithme qui permet de calculer la structure d'anneau de $\mathsf{h}(G/B)$ et d'y calculer les classes de variétés de Schubert désingularisées et leur produits.

1. Introduction

Let H be an algebraic cohomology theory endowed with Chern classes c_i such that for any two line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ over a variety X we have

$$c_1(\mathcal{L}_1 \otimes \mathcal{L}_2) = c_1(\mathcal{L}_1) + c_2(\mathcal{L}_2). \tag{1}$$

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The basic example of such a theory is the Chow group CH of algebraic cycles modulo rational equivalence.

Let G be a split semi-simple linear algebraic group over a field k and let T be a split maximal torus inside G contained in a Borel subgroup B. Consider the variety G/B of Borel subgroups of G with respect to T. In two classical papers [7] and [8] Demazure studied the cohomology ring $\mathsf{H}(G/B;\mathbb{Z})$ and provided an algorithm to compute $\mathsf{H}(G/B;\mathbb{Z})$ in terms of generators and relations.

The main object of his consideration was the so called characteristic map

$$\mathfrak{c}\colon S^*(M) \to \mathsf{H}(G/B;\mathbb{Z}), \tag{2}$$

where $S^*(M)$ is the symmetric algebra of the character group M of T. In [7] Demazure interpreted this map from the point of view of invariant theory of the Weyl group W of G by identifying its kernel with the ideal generated by non-constant invariants $S^*(M)^W$. The cohomology ring $\mathsf{H}(G/B;\mathbb{Z})$ was then replaced by a certain algebra constructed in terms of operators and defined in purely combinatorial terms.

In the present paper, we generalize most of the results of [7] to the case of an arbitrary oriented cohomology theory h, i.e., when (1) is replaced by

$$c_1(\mathscr{L}_1 \otimes \mathscr{L}_2) = F(c_1(\mathscr{L}_1), c_1(\mathscr{L}_2))$$

where F is the formal group law associated to h. Such theories were extensively studied by Levine-Morel [14], Panin-Smirnov [16], Merkurjev [15] and others. Apart from the Chow ring, other examples include algebraic K-theory, étale cohomology $H^*_{\text{ét}}(-,\mu_m)$, $(m, \operatorname{char}(k)) = 1$, connective K-theory, elliptic cohomology and the universal such theory: algebraic cobordism Ω.

To generalize the characteristic map (2), we first introduce a substitute for the symmetric algebra $S^*(M) = \mathsf{CH}^*(BT)$. This new combinatorial object, which we call a *formal group ring*, is denoted by $R[\![M]\!]_F$, where $R = \mathsf{h}(pt)$ is the coefficient ring, and can be viewed as a substitute of the cohomology ring of the classifying space $\mathsf{h}(BT)$ of T. In the case of a finite group G this object is related to generalized group characters studied in [12]. As in [7], we introduce a subalgebra $\mathscr{D}(M)_F$ of the R-linear endomorphisms of $R[\![M]\!]_F$ generated by specific differential operators and by taking its R-dual we obtain $\mathscr{H}(M)_F$, a combinatorial substitute for the cohomology ring $\mathsf{h}(G/B;\mathbb{Z})$. The characteristic map (2) then turns into the map

$$\mathfrak{c}\colon R[\![M]\!]_F \to \mathscr{H}(M)_F.$$

The Weyl group W acts naturally on $R[\![M]\!]_F$ and the main result of our paper (Theorems 6.4 and 6.9) says that:

THEOREM. – *If the torsion index of G is invertible in R and R has no 2-torsion, then the characteristic map is surjective and its kernel is generated by W-invariant elements in the augmentation ideal.*

Demazure's methods to prove this theorem in the special case of the additive formal group law do not work in general, for the following reason: the main objects used in his proofs are operators Δ_w for every $w \in W$. They are defined first for simple reflections, and afterwards for any w by decomposing it into simple reflections and composing the corresponding operators. It is then proved that the resulting composition is independent of the decomposition of w. For more general formal group laws, similar operators can still be defined for simple reflections (see Definition 3.5), but independence of the decomposition does not hold, as it was observed in [5]. Geometrically, it can be translated into the fact that the cobordism class of a desingularized Schubert variety depends on the desingularization, and not only on the Schubert variety itself (see Lemma 13.3). We overcome this problem by working with suitable filtrations such that the associated graded structures are covered by the additive case of Demazure (see in particular Proposition 4.4). We therefore encourage the reader unfamiliar with [7] to start by having a quick look at it before reading our Sections 3 to 7.

As an immediate application of the developed techniques, we provide an efficient algorithm for computing the cohomology ring $\mathcal{H}(M)_F = \mathsf{h}(G/B;\mathbb{Z})$. To do this we generalize the Bott-Samelson approach introduced in [1] and [8]. For oriented topological theories, some algorithms were considered by Bressler-Evens in [4, 5] and for algebraic theories in characteristic 0 by Hornbostel-Kiritchenko in [13]. See Remark 15.2 for a comparison.

Note that the theorem also provides another approach to computing the cohomology ring $\mathsf{h}(G/B;\mathbb{Z})$ by looking at the subring of invariants $R[\![M]\!]_F^W$. Observe that in the classical case when $\mathsf{h} = \mathsf{CH}$ (or K_0) and G is simply-connected it is known that $R[\![M]\!]_F^W$ is a power series ring in basic polynomial invariants (resp. fundamental representations). In general, the structure of $R[\![M]\!]_F^W$ remains unknown.

Finally, for the reader primarily interested in topology, let us mention that while all our proofs are algebraic and written in the language of algebraic geometry, the results apply as they are to topological cobordism or other complex oriented theories. Indeed, there is a canonical ring morphism

$$\Omega^*(G/B) \to \mathrm{MU}^*(G/B(\mathbb{C}))$$

(see [14, Ex. 1.2.10]) that is an isomorphism because both are free modules over the Lazard ring with bases corresponding to each other (given by desingularized Schubert cells).

The paper consists of three parts.

In the first part, we generalize the results of [7] by introducing and studying the generalized characteristic map $\mathfrak{c}\colon R[\![M]\!]_F \to \mathcal{H}(M)_F$. In Section 2, we introduce the formal group ring $R[\![M]\!]_F$ and prove its main properties. In Section 3, we define the main operators Δ and C on $R[\![M]\!]_F$. In Sections 4 and 5 we study the subalgebra $\mathcal{D}(M)_F$ of operators generated by Δ (resp. C) and multiplications. In Section 6 we define $\mathcal{H}(M)_F$ and prove the main theorem. In Section 7 we introduce a product on $\mathcal{H}(M)_F$ compatible with the characteristic map $\mathfrak{c}$.

In the second part, we generalize some of the results of [8] to arbitrary oriented cohomology theories. In Section 8, we discuss properties of oriented theories. In Sections 9 to 12, we carry out the Bott-Samelson desingularization approach.

In the last part, we apply the results of the first and the second parts to obtain information about the ring structure in $\mathsf{h}(G/B;\mathbb{Z})$. In Section 13, we prove that our algebraic replacement $\mathcal{H}(M)_F$ is isomorphic (as a ring) to the oriented cohomology $\mathsf{h}(G/B;\mathbb{Z})$ and that the characteristic maps $\mathfrak{c}_{G/B}$ and $\mathfrak{c}$ correspond to each other via this isomorphism. In Section 14, we give an algebraic description of the push-forward to the point and we prove various formulas. In Sections 15 and 16, we explain an algorithm to compute the ring structure of $\mathsf{h}(G/B;\mathbb{Z})$ and the Landweber-Novikov operations on algebraic cobordism. Finally, in Section 17, we give multiplication tables for $\Omega^*(G/B;\mathbb{Z})$ for groups G of rank 2.

Notation

Let k denote a base field of arbitrary characteristic. A variety over k means a reduced irreducible scheme of finite type over k. By X and Y we always mean smooth varieties over k. The base point Spec k is denoted by pt.

A ring always means a commutative ring with a unit and R always denotes a ring. A ring R' is called an R-algebra if it comes equipped with an injective ring homomorphism $R \hookrightarrow R'$. The letter M always denotes an Abelian group.

All formal group laws are assumed to be one-dimensional and commutative. Let F denote a formal group law and let $\mathbb{L}$ denote the Lazard ring, i.e., the coefficient ring of the universal formal group law U.

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PART I

INVARIANTS, TORSION INDICES AND FORMAL GROUP LAWS

2. Formal group rings

Let R be a ring, let M be an Abelian group and let F be a formal group law over R. In the present section we introduce and study the *formal group ring* $R[\![M]\!]_F$. For this, we use several auxiliary facts concerning topological rings and their completions which can be found in [3, III, §2]. The main result here is the decomposition Theorem 2.11. At the end we provide some examples of computations of $R[\![M]\!]_F$.

Definition 2.1. – Let R be a ring and let S be a set. Let $R[x_S] := R[x_s, s \in S]$ denote the polynomial ring over R with variables indexed by S. Let $\epsilon : R[x_S] \to R$ be the augmentation morphism which maps any x_s to 0. Consider the $\ker(\epsilon)$-adic topology on $R[x_S]$ given by ideals $\ker(\epsilon)^i$, $i \geq 0$, which form a fundamental system of open neighborhoods of 0. Note that a polynomial is in $\ker(\epsilon)^i$ if and only if its valuation is at least i, hence, we have $\cap_i \ker(\epsilon)^i = \{0\}$ and the $\ker(\epsilon)$-adic topology is Hausdorff.

We define $R[\![x_S]\!]$ to be the $\ker(\epsilon)$-adic completion of the polynomial ring $R[x_S]$.