ISSN 0012-9593

quatrième série - tome 46 fascicule 3 mai-juin 2013

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

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SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Ann. Scient. Éc. Norm. Sup.
4^{e} série, t. 46, 2013, p. 495 à 548

ON THE CONFORMAL GAUGE OF A COMPACT METRIC SPACE

BY Matias CARRASCO PIAGGIO

Abstract. – In this article we study the Ahlfors regular conformal gauge of a compact metric space (X, d), and its conformal dimension $\dim_{AR}(X, d)$. Using a sequence of finite coverings of (X, d), we construct distances in its Ahlfors regular conformal gauge of controlled Hausdorff dimension. We obtain in this way a combinatorial description, up to bi-Lipschitz homeomorphisms, of all the metrics in the gauge. We show how to compute $\dim_{AR}(X, d)$ using the critical exponent Q_N associated to the combinatorial modulus.

Résumé. – Dans cet article, on étudie la jauge conforme Ahlfors régulière d'un espace métrique compact et sa dimension conforme $\dim_{AR}(X, d)$. À l'aide d'une suite de recouvrements finis de (X, d), on construit des distances dans sa jauge Ahlfors régulière de dimension de Hausdorff contrôlée. On obtient ainsi une description combinatoire, à homéomorphismes bi-Lipschitz près, de toutes les métriques dans la jauge. On montre comment calculer $\dim_{AR} X$ à partir de modules combinatoires en considérant un exposant critique Q_N.

1. Introduction

The subject of this article is the study of quasisymmetric deformations of a compact metric space. More precisely, let (X, d) be a compact metric space, we are interested in its *conformal gauge*:

$$\mathscr{J}(X, d) := \{\theta \text{ distance on } X : \theta \sim_{qs} d\},$$

where two distances in X, d and θ, are quasisymmetrically equivalent $d \sim_{qs} \theta$ if the identity map $id : (X, d) \to (X, \theta)$ is a quasisymmetric homeomorphism. Recall that a homeomorphism $h : (X, d) \to (Y, \theta)$ between two metric spaces is *quasisymmetric* if there is an increasing homeomorphism $\eta : \mathbb{R}_+ \to \mathbb{R}_+$—called a distortion function—such that:

$$\frac{\theta(h(x), h(z))}{\theta(h(y), h(z))} \leq \eta\left(\frac{d(x, z)}{d(y, z)}\right),$$

for all $x, y, z \in X$ with $y \neq z$. In other words, a homeomorphism is quasisymmetric if it distorts relative distances in a uniform and scale invariant fashion. This class of maps provides a natural substitute of quasiconformal homeomorphisms in the broader context of metric spaces. Their precise definition was given by Tukia and Väisälä in [29]. See [21] for a detailed exposition of these notions.

For example, if d is a distance in X, then d^ϵ is also a distance for all $\epsilon \in (0,1]$, and the identity map $id : (X,d) \to (X,d^\epsilon)$ is η-quasisymmetric with $\eta(t) = t^\epsilon$. In particular, $\dim_H(X,d^\epsilon) = \epsilon^{-1}\dim_H(X,d)$. Therefore, quasisymmetric homeomorphisms can distort the Hausdorff dimension of the space, and one can always find distances in the gauge of arbitrarily large dimension.

The conformal gauge encodes the quasisymmetric invariant properties of the space. A fundamental quasisymmetry numerical invariant is the conformal dimension introduced by P. Pansu in [27]. There are different related versions of this invariant; in this article we are concerned with the *Ahlfors regular conformal dimension*, which is a variant introduced by M. Bourdon and H. Pajot in [8].

A distance $\theta \in \mathscr{J}(X,d)$ is Ahlfors regular of dimension $\alpha > 0$—AR for short—if there exist a Radon measure μ on X and a constant $K \geq 1$ such that:

$$K^{-1} \leq \frac{\mu(B_r)}{r^\alpha} \leq K,$$

for any ball B_r of radius $0 < r \leq \operatorname{diam}_\theta X$. In that case, μ is comparable to the α-dimensional Hausdorff measure and $\alpha = \dim_H(X,\theta)$ is the Hausdorff dimension of (X,θ). The collection of all AR distances in $\mathscr{J}(X,d)$ is the *Ahlfors regular conformal gauge* of (X,d), and is denoted by $\mathscr{J}_{AR}(X,d)$.

The AR conformal dimension measures the simplest representative of the gauge. It is defined by

$$\dim_{AR}(X,d) := \inf\left\{\dim_H(X,\theta) : \theta \in \mathscr{J}_{AR}(X,d)\right\}.$$

We write $\dim_{AR} X$ when there is no ambiguity on the metric d. Note that we always have the estimate $\dim_T X \leq \dim_{AR} X$, where $\dim_T X$ denotes the topological dimension of X. Apart from this, the AR conformal dimension is generally difficult to estimate. However, it was computed by P. Pansu for the boundaries of homogeneous spaces of negative curvature [27]. An exposition of the theory of conformal dimension, its variants and its applications can be found in [25], [2], [23], [18] and [26].

The interest in studying quasisymmetric invariants comes from the strong relationship between the geometric properties of a Gromov-hyperbolic space and the analytical properties of its boundary at infinity. Quasi-isometries between hyperbolic spaces induce quasisymmetric homeomorphisms between their boundaries, so any quasisymmetric invariant gives a quasi-isometric one.

For hyperbolic groups, the understanding of the canonical conformal gauge of the boundary at infinity—induced by the visual metrics—is an important step in the approach by Bonk and Kleiner to the characterization problem of uniform lattices of $\mathrm{PSL}_2(\mathbb{C})$, via their boundaries—Cannon's conjecture [5]. They showed that Cannon's conjecture is equivalent to the following: if G is a hyperbolic group, whose boundary is homeomorphic to the topological two-sphere S^2, then the Ahlfors regular conformal dimension of ∂G is attained. Motivated by Sullivan's dictionary, Haïssinsky and Pilgrim translated these notions to the context of branched coverings [20]. In particular, the AR conformal dimension characterizes rational maps between CXC branched coverings (see [20]).

Discretization has proved to be a useful tool in the study of conformal analytical objects in metric spaces. Different versions of combinatorial modulus have been considered by several authors, in connection with Cannon's conjecture (see [9, 4, 17]). The combinatorial modulus is a discrete version of the analytical conformal modulus from complex analysis, but unlike the latter, is independent of any analytical framework. It is defined using coverings of X; therefore, it depends only on the combinatorics of such coverings. In [6], the authors proved several important properties of combinatorial modulus for approximately self-similar sets. By defining a combinatorial modulus of a metric space (X, d) that takes into account all the "annuli" of the space, with some fixed radius ratio, we extend to a more general setting some of these properties.

The two main results of the present paper are Theorem 1.1 and Theorem 1.3. The first gives a combinatorial description of the AR conformal gauge from an appropriate sequence of coverings of the space. The second shows how to compute the AR conformal dimension using a critical exponent associated to the combinatorial modulus. The main technical result of the paper is Theorem 1.2, which gives sufficient conditions to bound from above the AR conformal dimension of X. To state the theorems we need to introduce some definitions.

Given an appropriate sequence of finite coverings $\{\mathcal{S}_n\}_n$ of X, with

$$||\mathcal{S}_n|| := \max\{\operatorname{diam} B : B \in \mathcal{S}_n\} \to 0, \text{ as } n \to +\infty, \tag{1.1}$$

we adapt a construction of Elek, Bourdon and Pajot [8, 15], and construct a geodesic hyperbolic metric graph Z_d with boundary at infinity homeomorphic to X (see Section 2 for precise definitions). With this identification the distance d becomes a visual metric on ∂Z_d. The vertices of the graph Z_d are the elements of $\mathcal{S} := \bigcup_n \mathcal{S}_n$, and the edges are of two types: vertical or horizontal. The vertical edges form a connected rooted tree T—which is a spanning tree of Z_d—and the horizontal ones describe the combinatorics of intersections of the elements of $\mathcal{S}$, i.e., two vertices B and B' in the same $\mathcal{S}_n$ are connected by an edge if $\lambda \cdot B \cap \lambda \cdot B' \neq \varnothing$, where λ is a large enough universal constant. We remark that one of the assumptions involving the elements of $\mathcal{S}$ is that they are "almost balls" (see (2.1),(2.2)). In particular, it makes sense to write $\lambda \cdot B$, and to talk about the center of B, for an element $B \in \mathcal{S}$ (see Section 2).

The vertical edges connect an element of $\mathcal{S}_n$ with an element of $\mathcal{S}_m$ for $|n - m| = 1$. All the edges of Z_d are isometric to the unit interval $[0, 1]$. We denote by w the root of T, and $B \sim B'$ means that B and B' are connected by a horizontal edge. For each $n \geq 0$, we denote by G_n the subgraph of Z_d consisting of all the vertices in $\mathcal{S}_n$ with all the horizontal edges of Z_d connecting two of them.

Consider a function $\rho : \mathcal{S} \to (0, 1)$. This function can be interpreted as an assignment of "new relative radius" of the elements of $\mathcal{S}$, or as an assignment of "new lengths" for the edges of Z_d. For each element $B \in \mathcal{S}$, there exists a unique geodesic segment in Z_d which joins the base point w and B; it consists of vertical edges and we denote it by $[w, B]$. The "new radius" of an element $B \in \mathcal{S}$ is expressed by the function $\pi : \mathcal{S} \to (0, 1)$ given by

$$\pi(B) := \prod \rho(B'),$$

where the product is taken over all elements $B' \in \mathcal{S} \cap [w, B]$. Theorem 1.1 says that from an appropriate function $\rho : \mathcal{S} \to (0,1)$ one can change the lengths of the edges of Z_d, and obtain a metric graph Z_ρ quasi-isometric to Z_d. This graph admits a visual metric θ_ρ, automatically in $\mathcal{J}_{AR}(X,d)$, of controlled Hausdorff dimension. When ρ goes through all the possible choices we get all the gauge $\mathcal{J}_{AR}(X,d)$ up to bi-Lipschitz homeomorphisms.

To state the conditions on the function ρ we need the following notation (see Section 2). For a path of edges in Z_d, $\gamma = \{(B_i, B_{i+1})\}_{i=1}^{N-1}$ with $B_i \in \mathcal{S}$, we define the ρ-length by

$$L_\rho(\gamma) = \sum_{i=1}^{N} \pi(B_i).$$

Let $\alpha > 1$. For $x, y \in X$, by the assumption (1.1), there exists a maximal level $m \in \mathbb{N}$ with the property that there exists an element $B \in \mathcal{S}_m$ with $x, y \in \alpha \cdot B$. We let

$$c_\alpha(x,y) := \{B \in S_m : x, y \in \alpha \cdot B\},$$

and we call it *the center* of x and y. We define $\pi(c_\alpha(x,y))$ as the maximum of $\pi(B)$ for $B \in c_\alpha(x,y)$. We also define $\Gamma_n(x,y)$ as the family of paths in Z_d that join two elements B and B' of $\mathcal{S}_n$, with $x \in B$ and $y \in B'$. We remark that the paths in $\Gamma_n(x,y)$ are not constrained to be contained in G_n. Finally, for an element $B \in \mathcal{S}_m$ and $n > m$, we denote by $D_n(B)$ the set of elements B' in $\mathcal{S}_n$ such that the geodesic segment $[w, B']$ contains B.

The conditions to be imposed to the wight function ρ are the following:

(H1) (Quasi-isometry) There exist $0 < \eta_- \leq \eta_+ < 1$ so that $\eta_- \leq \rho(B) \leq \eta_+$ for all $B \in \mathcal{S}$.

(H2) (Gromov product) There exists a constant $K_0 \geq 1$ such that for all $B, B' \in \mathcal{S}$ with $B \sim B'$, we have

$$\frac{\pi(B)}{\pi(B')} \leq K_0.$$

(H3) (Visual parameter) There exist $\alpha \in [2, \lambda/4]$ and a constant $K_1 \geq 1$ such that for any pair of points $x, y \in X$, there exists $n_0 \geq 1$ such that if $n \geq n_0$ and γ is a path in $\Gamma_n(x,y)$, then

$$L_\rho(\gamma) \geq K_1^{-1} \cdot \pi(c_\alpha(x,y)).$$

(H4) (Ahlfors regularity) There exist $p > 0$ and a constant $K_2 \geq 1$ such that for all $B \in \mathcal{S}_m$ and $n > m$, we have

$$K_2^{-1} \cdot \pi(B)^p \leq \sum_{B' \in D_n(B)} \pi(B')^p \leq K_2 \cdot \pi(B)^p.$$

We obtain the following results.

THEOREM 1.1 (Combinatorial description of the gauge). – *Let (X,d) be a compact metric space such that $\mathcal{J}_{AR}(X,d) \neq \varnothing$. Suppose the function $\rho : \mathcal{S} \to (0,1)$ verifies the conditions (H1), (H2), (H3) and (H4). Then there exists a distance θ_ρ on X quasisymmetrically equivalent to d and Ahlfors regular of dimension p. Furthermore, the distortion function of $\mathrm{id} : (X,d) \to (X,\theta_\rho)$ depends only on the constants η_-, η_+, K_0 and K_1, and*

$$\theta(x,y) \asymp \pi(c_\alpha(x,y)),$$

for all points $x, y \in X$. Conversely, any distance in the AR conformal gauge of (X,d) is bi-Lipschitz equivalent to a distance built in that way.