ISSN 0012-9593

quatrième série - tome 46 *fascicule 4* *juillet-août 2013*

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Jun-Muk HWANG

Varieties of minimal rational tangents of codimension 1

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Ann. Scient. Éc. Norm. Sup.
4[e] série, t. 46, 2013, p. 629 à 649

VARIETIES OF MINIMAL RATIONAL TANGENTS OF CODIMENSION 1

BY JUN-MUK HWANG

ABSTRACT. – Let X be a uniruled projective manifold and let x be a general point. The main result of [2] says that if the $(-K_X)$-degrees (i.e., the degrees with respect to the anti-canonical bundle of X) of all rational curves through x are at least $\dim X + 1$, then X is a projective space. In this paper, we study the structure of X when the $(-K_X)$-degrees of all rational curves through x are at least $\dim X$.

Our study uses the projective variety $\mathscr{C}_x \subset \mathbb{P}T_x(X)$, called the VMRT at x, defined as the union of tangent directions to the rational curves through x with minimal $(-K_X)$-degree. When the minimal $(-K_X)$-degree of rational curves through x is equal to $\dim X$, the VMRT $\mathscr{C}_x$ is a hypersurface in $\mathbb{P}T_x(X)$. Our main result says that if the VMRT at a general point of a uniruled projective manifold X of dimension ≥ 4 is a smooth hypersurface, then X is birational to the quotient of an explicit rational variety by a finite group action. As an application, we show that, if furthermore X has Picard number 1, then X is biregular to a hyperquadric.

RÉSUMÉ. – Soit X une variété projective uniréglée et soit x un point général. D'après le résultat principal de [2], si le degré par rapport à $-K_X$ de toute courbe rationnelle passant par x est au moins égal à $\dim(X) + 1$, alors X est un espace projectif. Dans cet article, nous étudions la structure de X sous l'hypothèse que le degré par rapport à $-K_X$ de toute courbe rationnelle passant par x est au moins égal à $\dim(X)$.

Notre étude repose sur la variété projective $\mathscr{C}_x \subset \mathbb{P}T_x(X)$ que nous appelons la VMRT (variété des tangentes des courbes rationnelles minimales) en x et qui est définie comme la réunion de toutes les directions tangentes aux courbes rationnelles passant par x dont le degré par rapport à $-K_X$ est minimal. Lorsque ce degré est égal à $\dim(X)$, la VMRT $\mathscr{C}_x$ est une hypersurface de $\mathbb{P}T_x(X)$. Notre résultat principal affirme que si la VMRT en un point général d'une variété projective uniréglée X de dimension $\geq$ 4 est une hypersurface, alors X est birationnelle au quotient d'une variété rationnelle explicite par l'action d'un groupe fini. Si, de plus, le rang du groupe de Picard de X est égal à 1, nous en déduisons que X est une hypersurface quadrique d'un espace projectif.

Supported by National Researcher Program 2010-0020413 of NRF and MEST.

1. Introduction and statement of main results

We will work over the complex numbers. For a projective manifold X, we have the notion of the (normalized) space $\mathrm{RatCurves}^{\mathrm{n}}(X)$ of rational curves on X (cf. [11] Section II.2 for the definition). When X is uniruled, this space $\mathrm{RatCurves}^{\mathrm{n}}(X)$ has an irreducible component $\mathscr{K}$ such that the subscheme $\mathscr{K}_x$ consisting of members of $\mathscr{K}$ through a general point $x \in X$ is non-empty. If furthermore $\mathscr{K}_x$ is projective, the component $\mathscr{K}$ is called a *family of minimal rational curves* and its members are called *minimal rational curves* of X. For a uniruled projective manifold X, a family of minimal rational curves always exists. For example, fix an ample line bundle L on X and pick a rational curve C through a general point of X which has minimal degree with respect to L. Then the component of $\mathrm{RatCurves}^{\mathrm{n}}(X)$ containing C is a family of minimal rational curves. Minimal rational curves play a crucial role in the geometry of X and have been much studied for that reason.

Given a family $\mathscr{K}$ of minimal rational curves, denote by $\deg(\mathscr{K})$ the degree of members of $\mathscr{K}$ with respect to the anti-canonical divisor $-K_X$. Then it is well-known that $\deg(\mathscr{K}) \leq \dim X + 1$ (e.g., [11] Corollary IV.1.15). When $\deg(\mathscr{K})$ is maximal, i.e., $\deg(\mathscr{K}) = \dim X + 1$, the biregular structure of X is completely known: Theorem 0.2 of [2] says that X is biregular to projective space and minimal rational curves are lines.

In this paper, we study the next-to-maximal case, i.e., when $\deg(\mathscr{K}) = \dim X$. To see what is to be expected, recall the following examples which are introduced in Example 1.7 of [5].

Example 1.1. – Let $Z \subset \mathbb{P}^{n-1}, n \geq 3$, be a submanifold. Regard Z as a submanifold of a hyperplane $\mathbb{P}^{n-1} \subset \mathbb{P}^n$. Let $\beta : X_Z \to \mathbb{P}^n$ be the blow-up of $\mathbb{P}^n$ along Z. Let $\mathscr{K}_Z$ be the component of $\mathrm{RatCurves}^{\mathrm{n}}(X_Z)$ determined by the proper transforms of lines on $\mathbb{P}^n$ intersecting Z. Then $\mathscr{K}_Z$ is a family of minimal rational curves on X_Z. If $Z \subset \mathbb{P}^{n-1}$ is a hypersurface, then $\deg(\mathscr{K}_Z) = n$.

Another example can be obtained by taking a cyclic quotient of Example 1.1:

Example 1.2. – In the setting of Example 1.1, let G be a finite group acting on $\mathbb{P}^n$ preserving the two submanifolds $Z \subset \mathbb{P}^{n-1} \subset \mathbb{P}^n$. The G-action can be lifted to a G-action on X_Z and the quotient X_Z/G exists as a normal variety. Let $\widetilde{X_Z/G}$ be a desingularization of X_Z/G which leaves the smooth locus intact. For some choice of Z and G, the family of rational curves given by the proper images of members of $\mathscr{K}_Z$ in $\widetilde{X_Z/G}$ becomes a family of minimal rational curves on $\widetilde{X_Z/G}$. For example, when Z is a Fermat hypersurface of large degree, we can find a cyclic group G acting on $\mathbb{P}^n$ preserving $Z \subset \mathbb{P}^{n-1}$ such that the action is free outside a finite set $F \subset \mathbb{P}^n$ with $F \cap Z = \varnothing$. When we regard F as a subset in X_Z, the G-action on X_Z is free outside $F \subset X_Z$ which is disjoint from members of $\mathscr{K}_Z$ through a general point. Thus the quotient $X_Z \to X_Z/G$ is unramified at all points of the members of $\mathscr{K}_Z$ through general points, which shows that $\mathscr{K}_Z$ descends to a family of minimal rational curves on $\widetilde{X_Z/G}$.

More examples can be constructed by suitable birational transformations of X_Z as follows.

EXAMPLE 1.3. – Given a projective manifold X and a family of minimal rational curves $\mathcal{K}$ with $\deg(\mathcal{K}) = \dim X$, there exists a non-trivial birational transformation $\Phi : X \dashrightarrow X'$ such that the proper images of members of $\mathcal{K}$ give rise to a family of minimal rational curves $\mathcal{K}'$ on X'. Then $\deg(\mathcal{K}') = \dim X'$. For example, for any finite subset $F \subset X$, all members of $\mathcal{K}$ through a general point $x \in X$ are disjoint from F. Thus the blow-up $\Phi : X \dashrightarrow \mathrm{Bl}_F(X)$ provides an example of such birational transformations.

These examples show that, unlike in the case of $\deg(\mathcal{K}) = \dim X + 1$, there is no simple biregular classification of X in the case of $\deg(\mathcal{K}) = \dim X$. What we can hope is to give a birational classification of X together with a description of $\mathcal{K}$. We achieve this for projective manifolds of dimension ≥ 4, modulo one technical assumption. To explain this technical assumption, let us recall the notion of the variety of minimal rational tangents (to be abbreviated as VMRT) associated to a family of minimal rational curves $\mathcal{K}$. The *VMRT* at a general point $x \in X$ is the subvariety $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ defined as the set of tangent directions at x of the members of $\mathcal{K}_x$ (cf. [4] Section 1.3). The dimension of $\mathcal{C}_x$ is equal to $\deg(\mathcal{K}) - 2$. The technical assumption we need is the following.

(*) The VMRT $\mathcal{C}_x$ at a general point $x \in X$ is smooth.

It has been conjectured that (*) is true for any X and any $\mathcal{K}$. The main result of [8] says that the normalization of $\mathcal{C}_x$ is smooth, so (*) is equivalent to the normality of $\mathcal{C}_x$. In specific problems involving minimal rational curves, the assumption (*) is often verifiable directly. For example, when $\deg(\mathcal{K}) = \dim X + 1$, $\mathcal{C}_x = \mathbb{P}T_x(X)$ is trivially smooth, which plays an implicit role in [2]. It is also easy to check (*) in Examples 1.1 and 1.2. In fact, $\mathcal{C}_x \subset \mathbb{P}T_x(X_Z)$ is isomorphic to $Z \subset \mathbb{P}^{n-1}$ as projective subvariety. In this sense, it is reasonable to keep (*) as a working hypothesis when studying minimal rational curves.

Our main result is the following. Note that under the assumption of (*) and $\deg(\mathcal{K}) = n$, the VMRT $\mathcal{C}_x$ is a smooth hypersurface in $\mathbb{P}T_x(X)$.

THEOREM 1.4. – *Let X be a uniruled projective manifold of dimension $n \geq 4$ with a family $\mathcal{K}$ of minimal rational curves such that the VMRT $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ at a general point $x \in X$ is a smooth hypersurface of degree $m \geq 3$. Then there exist a smooth hypersurface $Z \subset \mathbb{P}^{n-1}$ of degree m, a finite group G acting on $\mathbb{P}^n$ preserving $Z \subset \mathbb{P}^{n-1}$ and a birational map $\Phi : X_Z/G \dashrightarrow X$ such that the dominant rational map $\Psi : X_Z \dashrightarrow X$ defined by the composition of Φ and the quotient $X_Z \to X_Z/G$ sends general members of $\mathcal{K}_Z$ to those of $\mathcal{K}$.*

The assumption $m \geq 3$ in Theorem 1.4 is harmless. When the degree m of the hypersurface $\mathcal{C}_x$ is 1 or 2, the structure of X is already well-understood. When $m = 1$, Theorem 1.1 of [1] says that there exist a Zariski open subset $X^o \subset X$ and a $\mathbb{P}^{n-1}$-fibration $X^o \to B$ to a (quasi-projective) curve B such that members of $\mathcal{K}$ correspond to lines in the $\mathbb{P}^{n-1}$-fibers. When $m = 2$, i.e., $\mathcal{C}_x$ is a hyperquadric, the conclusion of Theorem 1.4 follows from the work of Mok ([13]) as a variation of its Main Theorem. Strictly speaking, [13] is written under the assumption that X has Picard number 1 and its Main Theorem is stated with that assumption. However, those arguments in [13] that are independent from that assumption and our argument in Section 5 provide the proof we need.

Theorem 1.4 gives a reasonably satisfactory classification under the technical assumption (*). However, to be a complete classification, we need to classify the pairs (Z, G) which actually occur and determine the birational type of X_Z/G. This seems to require a substantial work of quite different nature from the main theme of the current paper. Thus we will not touch upon these questions in the current paper and leave them for future research of interested readers.

Our approach to Theorem 1.4 is based on the study of geometric structures arising from minimal rational curves. To fix the terminology, we give the definition here.

Definition 1.5. – On a uniruled projective manifold X with a family $\mathscr{K}$ of minimal rational curves, the *VMRT-structure* $\mathscr{C} \subset \mathbb{P}T(X)$ is the irreducible subvariety defined by the closure of the union of the VMRT's $\mathscr{C}_x \subset \mathbb{P}T_x(X)$ as x varies over general points of X.

In contrast to previous terminology in [4], [8], etc., the term 'structure' is used here to emphasize our view-point that this subvariety should be viewed as a differential geometric object. In fact, a key notion is the equivalence of VMRT-structures in the following sense.

Definition 1.6. – Given two uniruled projective manifolds X^1 and X^2 with families of minimal rational curves $\mathscr{K}^1$ and $\mathscr{K}^2$ respectively, the corresponding VMRT-structure $\mathscr{C}^1 \subset \mathbb{P}T(X^1)$ at a point $x^1 \in X^1$ is *equivalent* to the VMRT-structure $\mathscr{C}^2 \subset \mathbb{P}T(X^2)$ at $x^2 \in X^2$, if there exist an analytic neighborhood $x^1 \subset U^1 \subset X^1$ (resp. $x^2 \subset U^2 \subset X^2$) and a biholomorphic map $\psi : U^1 \to U^2$ whose differential $d\psi : \mathbb{P}T(U^1) \to \mathbb{P}T(U^2)$ sends $\mathscr{C}^1 \cap \mathbb{P}T(U^1)$ to $\mathscr{C}^2 \cap \mathbb{P}T(U^2)$ biholomorphically.

To see the relevance of the equivalence of VMRT-structures in Theorem 1.4, recall the following result.

Theorem 1.7 (Main Theorem in [7]). – *Let X^1 (resp. X^2) be a uniruled projective manifold of Picard number 1, not biregular to projective space. Assume that X^1 (resp. X^2) has a family of minimal rational curves $\mathscr{K}^1$ (resp. $\mathscr{K}^2$) such that the VMRT $\mathscr{C}^1_{x_1} \subset \mathbb{P}T_{x_1}(X^1)$ at a general point $x^1 \in X^1$ (resp. $x^2 \in X^2$) is smooth and irreducible, and the VMRT-structure at x^1 is equivalent to the VMRT-structure at x^2 in the sense of Definition 1.6. Then the local equivalence map ψ in Definition 1.6 extends to a biregular morphism $\Psi : X^1 \to X^2$ which sends members of $\mathscr{K}^1$ to members of $\mathscr{K}^2$.*

It turns out that we can use some analogue of Theorem 1.7 to reduce the proof of Theorem 1.4, to showing that the VMRT-structure of X at a general point is equivalent to that of X_Z in Example 1.1. The argument we need for this reduction is a variation of the proof of Theorem 1.7 in [7]. This will be explained in Section 5.

From this, we may say that the gist of the matter is to study the equivalence problem for the VMRT-structure in the setting of Theorem 1.4. In [5], this equivalence problem is studied and the following result is proved (Theorem 1.11 in [5]).

Theorem 1.8. – *Let X be a uniruled projective manifold of dimension $n \geq 3$ with a family $\mathscr{K}$ of minimal rational curves with* $\deg(\mathscr{K}) = n$. *Assume that for a general $x \in X$,*

(i) *the VMRT $\mathscr{C}_x \subset \mathbb{P}T_x(X)$ is smooth;*
(ii) *the hypersurface $\mathscr{C}_x \subset \mathbb{P}T_x(X)$ has degree $m \geq 4$; and*