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DIFFUSION FOR THE PERIODIC WIND-TREE MODEL

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ABSTRACT. – The periodic wind-tree model is an infinite billiard in the plane with identical rectangular scatterers placed at each integer point. We prove that independently of the size of scatters and generically with respect to the angle, the polynomial diffusion rate in this billiard is $2/3$.

RÉSUMÉ. – Le vent dans les arbres périodique est un billard infini construit de la manière suivante. On considère le plan dans lequel sont placés des obstacles rectangulaires identiques à chaque point entier. Une particule (identifiée à un point) se déplace en ligne droite (le vent) et rebondit de manière élastique sur les obstacles (les arbres). Nous prouvons qu'indépendamment de la taille des obstacles et génériquement par rapport à l'angle initial de la particule le coefficient de diffusion polynomial des orbites de ce billard est $2/3$.

1. Introduction

The wind-tree model is a billiard in the plane introduced by P. Ehrenfest and T. Ehrenfest in 1912 ([7]). We study the periodic version studied by J. Hardy and J. Weber [14]. A point moves in the plane $\mathbb{R}^2$ and bounces elastically off rectangular scatterers following the usual law of reflection. The scatterers are translates of the rectangle $[0, a] \times [0, b]$ where $0 < a < 1$ and $0 < b < 1$, one centered at each point of $\mathbb{Z}^2$. We denote the complement of obstacles in the plane by $\mathrm{T}(a, b)$ and refer to it as the *wind-tree model* or the *infinite billiard table*. Our aim is to understand dynamical properties of the wind-tree model. We denote by $\phi_t^\theta : \mathrm{T}(a, b) \to \mathrm{T}(a, b)$ the billiard flow: for a point $p \in \mathrm{T}(a, b)$, the point $\phi_t^\theta(p)$ is the position of a particle after time t starting from position p in direction θ.

It is proved in [14] that the rate of diffusion in the periodic wind-tree model is $\log t \log\log t$ for very specific directions (generalized diagonals which correspond to angles of the form $\arctan(p/q)$ with $p/q \in \mathbb{Q}$). Their result was recently completed by J.-P. Conze and E. Gutkin [6] who explicit the ergodic decomposition of the billiard flow for those directions. K. Frączek and C. Ulcigrai recently proved that generically the billiard flow is non-ergodic. P. Hubert, S. Lelièvre and S. Troubetzkoy [17] proved that for a residual set of parameters a

and b, for almost every direction θ, the flow in direction θ is recurrent. In this paper, we compute the polynomial rate of diffusion of the orbits which is valid for almost every direction θ. We get the following.

THEOREM 1. – *Let $d(.,.)$ be the Euclidean distance on $\mathbb{R}^2$. Then for all parameters $(a,b) \in (0,1)^2$, Lebesgue-almost all θ and every point p in $\mathrm{T}(a,b)$ (with an infinite forward orbit)*

$$\limsup_{T\to+\infty} \frac{\log d(p,\phi_T^\theta(p))}{\log T} = \frac{2}{3}.$$

By the $\mathbb{Z}^2$-periodicity of the billiard table $\mathrm{T}(a,b)$, our problem reduces to understand deviations of a-$\mathbb{Z}^2$ cocycle over the billiard in a fundamental domain. On the other hand, as the barriers are horizontals and verticals, an orbit in $\mathrm{T}(a,b)$ with initial angle θ from the horizontal takes at most four different directions $\{\theta, \pi-\theta, -\theta, \pi+\theta\}$ (the billiard is rational). By a standard construction consisting of unfolding the trajectories [29], called the Katok-Zemliakov construction, the billiard flow can be replaced by a linear flow on a (non compact) translation surface which is made of four copies of $\mathrm{T}(a,b)$ that we denote $\mathrm{X}_\infty(a,b)$ (see Section 3.2 for the construction). The surface $\mathrm{X}_\infty(a,b)$ is $\mathbb{Z}^2$-periodic and we denote $\mathrm{X}(a,b)$ the quotient of $\mathrm{X}_\infty(a,b)$ under the $\mathbb{Z}^2$ action. As the unfolding procedure of the billiard flow is equivariant with respect to the $\mathbb{Z}^2$ action, $\mathrm{X}(a,b)$ can also be seen as the unfolding of the billiard in a fundamental domain of the action of $\mathbb{Z}^2$ on the billiard table $\mathrm{T}(a,b)$.

The position of the particle in $\mathrm{X}_\infty(a,b)$ can be tracked from $\mathrm{X}(a,b)$. More precisely, the position of the particle starting from $p \in \mathrm{X}_\infty(a,b)$ in direction θ can be approximated by the pairing of a geodesic $\gamma_t(p)$ of $\mathrm{X}(a,b)$ seen as an element of the homology with a cocycle $f \in H^1(\mathrm{X}(a,b);\mathbb{Z}^2)$ describing the infinite cover $\mathrm{X}_\infty(a,b) \to \mathrm{X}(a,b)$. The growth of pairing of a fixed cocycle with geodesics in a translation surface is equivalent to the growth of certain Birkhoff sums over an interval exchange transformation. The estimation can be obtained from the action of $\mathrm{SL}(2,\mathbb{R})$ on strata of translation surfaces $\mathcal{H}_g(\alpha)$ and more precisely of the Teichmüller flow which corresponds to the action of diagonal matrices $g_t = \left(\begin{smallmatrix} e^t & 0 \\ 0 & e^{-t} \end{smallmatrix}\right)$ (see Section 2 for precise definitions). As proved by A. Zorich [35, 36] the Kontsevich-Zorich cocycle over the Teichmüller flow can be used to estimate the deviations of Birkhoff sums for generic interval exchange transformations with respect to the Lebesgue measure. More precisely, he proved that the Lyapunov exponents of the Kontsevich-Zorich cocycle is the polynomial rate of deviations. G. Forni [12] relates this phenomenon to obstructions to solve cohomological equations and extends Zorich's proof to a more general context (see Section 9 of [12]).

The surface $\mathrm{X}(a,b)$ is a covering of the genus 2 surface $\mathrm{L}(a,b)$ which is a so called L-shaped surface that belongs to the stratum $\mathcal{H}(2)$. The orbit of $\mathrm{X}(a,b)$ for the Teichmüller flow belongs to a sub-locus of the moduli space $\mathcal{H}(2^4)$ that we call $\mathcal{G}$.

We now formulate a generalization of A. Zorich's and G. Forni's theorems about deviations of ergodic averages that is a central step in the proof of Theorem 1. Let $\mathcal{H}(\alpha)$ be a stratum of Abelian differentials and $Y \in \mathcal{H}(\alpha)$ a translation surface. The Teichmüller flow (g_t) can be used to renormalize the trajectories of the linear flow on Y. The Kontsevich-Zorich cocycle $B^{(t)}(Y) : H^1(Y;\mathbb{R}) \to H^1(g_t \cdot Y;\mathbb{R})$ (or KZ cocycle) measures the growth of cohomology vectors along the Teichmüller geodesic $(g_t \cdot Y)_t$. Let μ be a g_t-invariant ergodic probability measure on $\mathcal{H}(\alpha)$. It follows from [12], that the KZ cocycle is integrable for the measure μ. From Oseledets multiplicative ergodic theorem, there exist real numbers $\nu_1(\mu) > \nu_2(\mu) > \cdots > \nu_k(\mu) > 0$, such that for μ-almost every non zero Abelian differential $Y \in \mathcal{H}(\alpha)$ there exists a unique flag

$$H^1(Y;\mathbb{R}) = F_1^u \supset F_2^u \supset \cdots \supset F_k^u \supset F_{k+1}^u = F^c \supset F_k^s \supset \cdots \supset F_1^s \supset F_0^s = \{0\}$$

such that for any norm $\|.\|$ on $H^1(Y;\mathbb{R})$, for all $1 \le i \le k$

1. if $f \in F_i^u \setminus F_{i+1}^u$, then
$$\lim_{t\to\infty} \frac{\log \|B^{(t)}(Y)\cdot f\|}{\log t} = \nu_i(\mu),$$
2. if $f \in F_i^s \setminus F_{i-1}^s$, then
$$\lim_{t\to\infty} \frac{\log \|B^{(t)}(Y)\cdot f\|}{\log t} = -\nu_i(\mu),$$
3. if $f \in F^c \setminus F_k^s$, then
$$\lim_{t\to\infty} \frac{\log \|B^{(t)}(Y)\cdot f\|}{\log t} = 0.$$

There exist also positive integers m_i for $i = 1,\ldots,k$ and an integer m such that for μ almost all translation surface Y the filtration satisfies

– the dimension of F_i^s is $m_1 + \cdots + m_i$,
– the dimension of F^c is $m_1 + \cdots + m_k + 2m$,
– the dimension of F_i^u is $m_1 + \cdots + m_{i-1} + 2m_i + \cdots + 2m_k + 2m$.

From the definition of the Teichmüller flow and the KZ cocycle, it follows that $\nu_1 = 1$. Forni proved that $m_1 = 1$ [12]. The *Lyapunov spectrum of the KZ* cocycle is the multiset of numbers

$$\nu_1 = 1\ \underbrace{\nu_2\ldots\nu_2}_{m_2 \text{ times}}\ \ldots\ \underbrace{\nu_k\ldots\nu_k}_{m_k \text{ times}}\ \underbrace{0\ldots0}_{2m \text{ times}}\ \underbrace{-\nu_k\ldots-\nu_k}_{m_k \text{ times}}\ \ldots\ \underbrace{-\nu_2\ldots-\nu_2}_{m_2 \text{ times}}\ -1 = -\nu_1.$$

The numbers $\nu_i(\mu)$ for $i = 1,\ldots,k$ are called the *positive Lyapunov exponents (with respect to μ)*. The subspace $F^s = F_k^s$ is called the *stable space* (at Y) of the KZ cocycle.

In order to state a precise statement for deviations, one needs genericity with respect to Lyapunov exponents but also an extra assumption on recurrence. Let μ be a g_t ergodic measure on some stratum $\mathcal{H}(\alpha)$. We say that a surface $Y \in \mathcal{H}(\alpha)$ is *generic recurrent for μ* if there exist compact neighborhoods $U_i \subset \mathcal{H}(\alpha)$ of Y such that $\bigcap_i U_i = \{Y\}$ and

$$\lim_{t\to\infty} \frac{Leb(\{s; s\in[0,t] \quad \text{and} \quad g_sY \in U\})}{t} = \mu(U).$$

Birkhoff theorem ensures that this condition is satisfied for almost every surface.

THEOREM 2. – *Let μ be a g_t-ergodic measure on a stratum of Abelian differentials. Let ν_i for $i = 1,\ldots,k$ denote the positive Lyapunov exponents of the KZ cocycle for μ and denote, for an Oseledets generic surface Y, $F_i^u(Y)$, $F^c(Y)$ and $F_i^s(Y)$ the components of the flag of the Oseledets decomposition.*

Then, for a surface $Y \in \mathcal{H}(\alpha)$ which is generically recurrent and Oseledets generic for μ, for every point $p \in Y$ with an infinite forward orbit

1. *along the unstable space the growth is polynomial: for all* $1 \leq i \leq k$*, for all* $f \in F_i^u \setminus F_{i+1}^u$

$$\limsup_{T \to \infty} \frac{\log |\langle f, \gamma_T(p) \rangle|}{\log T} = \nu_i,$$

2. *along the central space the growth is sub-polynomial: for all* $f \in F^c \setminus F_k^s$

$$\limsup_{T \to \infty} \frac{\log |\langle f, \gamma_T(p) \rangle|}{\log T} = 0,$$

3. *along the stable space the growth is bounded: there exists a constant* C *such that for all* $f \in F^s$

$$\forall T \geq 0, \ |\langle f, \gamma_T(p) \rangle| \leq C \|f\|.$$

Theorem 2 has first been proved by A. Zorich [34, 35, 36] for the Lebesgue measure on a connected component of a stratum or equivalently for a generic interval exchange transformation. G. Forni [12] extended the theorem for a very large class of functions and for certain measures. More precisely, his proof of the lower bound relies on the existence of a particular translation surface in the support of the measure. A. Bufetov [3] gave a proof of Case 1 of Theorem 2 (when the cocycle f is associated with a positive Lyapunov exponent) in the general context of symbolic dynamics which applies in particular to translation flows (Propositions 2 and 5 of [3]). Our approach uses Veech's zippered rectangles [31] and gives a concrete version of the renormalization process by the Teichmüller flow and the Kontsevich-Zorich cocycle in the flavor of [35, 36] and [12].

On the other hand, from results of A. Eskin, M. Kontsevich and A. Zorich [9] about sum of Lyapunov exponents in hyperelliptic loci, we deduce that the Lyapunov exponent for $\mathrm{X}(a, b)$ which controls the deviation in the wind-tree model equals $2/3$. The value $2/3$ comes from algebraic geometry. More precisely, it corresponds to the degree of a subbundle of the Hodge bundle over the moduli space of complex curves (or Riemann surfaces) in which the wind-tree cocycle belongs.

Using only Birkhoff and Oseledets theorems, one can prove that the conclusion of Theorem 1 holds for almost every parameters a, b. In order to obtain all parameters we use a recent result of J. Chaika and A. Eskin [5] which asserts that Birkhoff theorem for regular functions and Oseledets theorem for the Kontsevich-Zorich cocycle are more regular for $\mathrm{SL}(2, \mathbb{R})$-invariant measures: they hold for all surfaces in almost every directions. The work of Chaika and Eskin strongly relies on previous work of A. Eskin and M. Mirzhakani [10] and A. Eskin, M. Mirzakhani and M. Mohamadi [11] on $\mathrm{SL}(2, \mathbb{R})$-invariant measures on strata of Abelian differentials.

The paper is organized as follows. In Section 2 we introduce the tools from Teichmüller theory which are involved in our proof of Theorem 1. In Section 3, we detail the unfolding procedure and prove that the distance in Theorem 1 corresponds to a pairing between a geodesic in $\mathrm{X}(a, b)$ with an integer cocycle. Then we reformulate Theorem 1 in the language of translation surfaces (see Theorem 6). In Section 4 we compute the Lyapunov exponents relative to every measure on $\mathcal{H}(2^4)$ which is supported on the closure of the $\mathrm{SL}(2, \mathbb{R})$-orbit of a surface $\mathrm{X}(a, b)$. Section 5 is devoted to the proof of Theorem 2.