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TRANSFORMING METRICS ON A LINE BUNDLE TO THE OKOUNKOV BODY

BY DAVID WITT NYSTRÖM

ABSTRACT. – Let L be a big holomorphic line bundle on a complex projective manifold X. We show how to associate a convex function on the Okounkov body of L to any continuous metric ψ on L. We will call this the Chebyshev transform of ψ, denoted by $c[\psi]$. Our main theorem states that the difference of metric volume of L with respect to two metrics, a notion introduced by Berman-Boucksom, is equal to the integral over the Okounkov body of the difference of the Chebyshev transforms of the metrics. When the metrics have positive curvature the metric volume coincides with the Monge-Ampère energy, which is a well-known functional in Kähler-Einstein geometry and Arakelov geometry. We show that this can be seen as a generalization of classical results on Chebyshev constants and the Legendre transform of invariant metrics on toric manifolds. As an application we prove the differentiability of the metric volume in the cone of big metrized $\mathbb{R}$-divisors. This generalizes the result of Boucksom-Favre-Jonsson on the differentiability of the ordinary volume of big $\mathbb{R}$-divisors and the result of Berman-Boucksom on the differentiability of the metric volume when the underlying line bundle is fixed.

RÉSUMÉ. – Soit L un fibré en droite holomorphe gros sur une variété complexe X, projective et lisse. Nous montrons comment associer une fonction convexe sur le corps d'Okounkov de L à chaque métrique continue ψ sur L. Nous l'appelons la transformée de Chebyshev de ψ, désignée par $c[\psi]$. Notre théorème principal affirme que la différence des volumes métriques de L par rapport à deux métriques, une notion introduite par Berman-Boucksom, s'écrit comme une intégrale de la différence des transformations de Chebyshev des métriques. Quand les métriques sont de courbure positive, le volume métrique coïncide avec l'énergie de Monge-Ampère, qui est une fonctionnelle bien connue dans la géométrie de Kähler-Einstein et la géométrie d'Arakelov. On démontre que ceci peut être considéré comme une généralisation des résultats classiques sur les constantes de Chebyshev et la transformation de Legendre des métriques invariantes sur des variétés toriques. En guise d'application, on démontre la différentiabilité du volume métrique dans le cône des gros $\mathbb{R}$-diviseurs métrisés. Ceci généralise le résultat de Boucksom-Favre-Jonsson sur la différentiabilité du volume ordinaire des $\mathbb{R}$-diviseurs gros et le résultat de Berman-Boucksom sur la différentiabilité du volume métrique quand le fibré L est fixe.

1. Introduction

In [10] ([9] is a published shortened version) and [12] Kaveh-Khovanskii and Lazarsfeld-Mustață initiated a systematic study of Okounkov bodies of divisors and more generally of linear series. Our goal is to contribute with an analytic viewpoint.

It was Okounkov who in his papers [13] and [14] introduced a way of associating a convex body in $\mathbb{R}^n$ to any ample divisor on a n-dimensional projective variety. This convex body, called the Okounkov body of the divisor and denoted by $\Delta(L)$, can then be studied using convex geometry. It was recognized in [12] that the construction works for arbitrary big divisors.

We will restrict ourselves to a complex projective manifold X, and instead of divisors we will for the most part use the language of holomorphic line bundles. Because of this, in the construction of the Okounkov body, we prefer choosing local holomorphic coordinates instead of the equivalent use of a flag of subvarieties (see [12]). We use additive notation for line bundles, i.e., we will write kL instead of $L^{\otimes k}$ for the k-th tensor power of L. We will also use the additive notation for metrics. If h is a hermitian metric on a line bundle, we may write it as $h = e^{-\psi}$, and in this paper we will denote that metric by ψ. Thus if ψ is a metric on L, $k\psi$ is a metric on kL. The pair (L, ψ) of a line bundle L with a continuous metric ψ will be called a metrized line bundle.

The main motivation for studying Okounkov bodies has been their connection to the volume function on divisors. Recall that the volume of a line bundle L is defined as

$$\mathrm{vol}(L) := \limsup_{k\to\infty} \frac{n!}{k^n} \dim(H^0(kL)).$$

A line bundle is said to be big if it has positive volume. From here on, all line bundles L we consider will be assumed to be big. By Theorem A in [12], for any big line bundle L it holds that

$$\mathrm{vol}(L) = n!\mathrm{vol}_{\mathbb{R}^n}(\Delta(L)).$$

We are interested in studying certain functionals on the space of metrics on L that refine $\mathrm{vol}(L)$.

The notion of a metric volume of a metrized line bundle (L, ψ) was introduced by Berman-Boucksom in [1]. Given a metric ψ one has a natural norm on the spaces of holomorphic sections $H^0(kL)$, namely the supremum norm

$$||s||_{k\psi,\infty} := \sup\{|s(x)|e^{-k\psi(x)/2} : x \in X\}.$$

Let $\mathcal{B}^\infty(k\psi) \subseteq H^0(kL)$ be the unit ball with respect to this norm.

$H^0(kL)$ is a vector space, thus given a basis we can calculate the volume of $\mathcal{B}^\infty(k\psi)$ with respect to the associated Lebesgue measure. This will depend on the choice of basis, but given a reference metric φ one can compute the quotient

$$\frac{\mathrm{vol}(\mathcal{B}^\infty(k\psi))}{\mathrm{vol}(\mathcal{B}^\infty(k\varphi))}$$

and this quantity will be invariant under the change of basis. The k:th $\mathcal{L}$-bifunctional is defined as

$$\mathcal{L}_k(\psi, \varphi) := \frac{n!}{2k^{n+1}} \log\left(\frac{\mathrm{vol}(\mathcal{B}^\infty(k\psi))}{\mathrm{vol}(\mathcal{B}^\infty(k\varphi))}\right).$$

The metric volume of a metrized line bundle (L,ψ), denoted by $\mathrm{vol}(L,\psi,\varphi)$, is defined as the limit

$$\mathrm{vol}(L,\psi,\varphi) := \lim_{k\to\infty} \mathcal{L}_k(\psi,\varphi). \tag{1}$$

Remark 1.1. – *In* [1] *this quantity is called the* energy at equilibrium*, but we have in this paper chosen to call it the metric volume in order to accentuate the close relationship with the ordinary volume of line bundles.*

The metric volume obviously depends on the choice of φ as a reference metric but it is easy to see that the difference of metric volumes $\mathrm{vol}(L,\psi,\varphi) - \mathrm{vol}(L,\psi',\varphi)$ is independent of the choice of reference.

The definition of the metric volume is clearly reminiscent of the definition of the volume of a line bundle. In fact, one easily checks that when adding 1 to the reference metric φ, we have that

$$\mathrm{vol}(L,\varphi+1,\varphi) = \mathrm{vol}(L).$$

From this it follows readily that the metric volume is zero whenever the line bundle fails to be big.

In [1] Berman-Boucksom prove that the limit (1) exists. They do this by proving that it actually converges to a certain integral over the space X involving mixed Monge-Ampère measures related to the metrics.

A metric ψ is said to be *psh* if the corresponding function expressed in a trivialization of the bundle is plurisubharmonic, so that

$$dd^c\psi \geq 0$$

as a current. Given two locally bounded psh metrics ψ and φ one defines $\mathcal{E}(\psi,\varphi)$ as

$$\frac{1}{n+1}\sum_{j=0}^{n}\int_X (\psi-\varphi)(dd^c\psi)^j \wedge (dd^c\varphi)^{n-j},$$

which we will refer to as the Monge-Ampère energy of ψ and φ. This bifunctional first appeared in the works of Mabuchi and Aubin in Kähler-Einstein geometry (see [1] and references therein).

If ψ and φ are continuous but not necessarily psh, we may still define a Monge-Ampère energy, by first projecting them down to the space of psh metrics,

$$P(\psi) := \sup\{\psi' : \psi' \leq \psi, \psi' \mathrm{psh}\},$$

and then integrating over the Zariski-open subset Ω where the projected metrics are locally bounded. We are therefore led to consider the composite functional $\mathcal{E}\circ P$:

$$\mathcal{E}\circ P(\psi,\varphi) := \frac{1}{n+1}\sum_{j=0}^{n}\int_\Omega (P(\psi)-P(\varphi))(dd^cP(\psi))^j \wedge (dd^cP(\varphi))^{n-j}. \tag{2}$$

The Monge-Ampère energy can also be seen as a generalization of the volume since if we let ψ be equal to $\varphi+1$, from e.g., [1] we have that

$$\mathcal{E}\circ P(\psi,\varphi) = \int_\Omega (dd^cP(\varphi))^n = \mathrm{vol}(L).$$

This is not a coincidence. In fact Berman-Boucksom prove that for any pair of continuous metrics ψ and φ on a big line bundle L we have that

$$\mathcal{E} \circ P(\psi, \varphi) = \operatorname{vol}(L, \psi, \varphi).$$

In [6] Boucksom-Favre-Jonsson proved that the volume function on the Néron-Severi space is $\mathscr{C}^1$ in the big cone. This result was later reproved in [12] by Lazarsfeld-Mustaţă using Okounkov bodies. Berman-Boucksom proved in [1] the differentiability of the metric volume when the line bundle is fixed. A natural question is what one can say about the regularity of the metric volume when the line bundle is allowed to vary as well. In this paper we approach this question by combining the pluripotential methods of Berman-Boucksom with Okounkov body techniques inspired by the work of Lazarsfeld-Mustaţă.

Given a continuous metric ψ, we will show how to construct an associated convex function on the interior of the Okounkov body of L which we will call the Chebyshev transform of ψ, denoted by $c[\psi]$. The construction can be seen to generalize both the Chebyshev constants in classical potential theory and the Legendre transform of convex functions (see Subsections 9.2 and 9.3 respectively).

First we describe how to construct $\Delta(L)$. Choose a point $p \in X$ and local holomorphic coordinates $z_1, \ldots, z_n$ centered at p. Choose also a trivialization of L around p. With respect to this trivialization any holomorphic section $s \in H^0(L)$ can be written as a convergent power series in the coordinates z_i,

$$s = \sum_{\alpha} a_\alpha z^\alpha.$$

Consider the lexicographic order on $\mathbb{N}^n$, and let $v(s)$ denote the smallest index α (i.e., with respect to the lexicographic order) such that

$$a_\alpha \neq 0.$$

We let $v(H^0(L))$ denote the set $\{v(s) : s \in H^0(L), s \neq 0\}$, and finally let the Okounkov body of L, denoted by $\Delta(L)$, be defined as closed convex hull in $\mathbb{R}^n$ of the union

$$\bigcup_{k \geq 1} \frac{1}{k} v(H^0(kL)).$$

Observe that the construction depends on the choice of p and the holomorphic coordinates. For other choices, the Okounkov bodies will in general differ.

Now let ψ be a continuous metric on L. There are associated supremum norms on the spaces of sections $H^0(kL)$,

$$||s||^2_{k\psi} := \sup_{x \in X} \{|s(x)|^2 e^{-k\psi(x)}\}.$$

If $v(s) = k\alpha$ for some section $s \in H^0(kL)$, we let $A_{\alpha,k}$ denote the affine space of sections in $H^0(kL)$ of the form

$$z^{k\alpha} + \text{ higher order terms.}$$

We define the discrete Chebyshev transform $F[\psi]$ on $\bigcup_{k \geq 1} v(H^0(kL)) \times \{k\}$ as

$$F[\psi](k\alpha, k) := \inf\{\ln ||s||^2_{k\psi} : s \in A_{\alpha,k}\}.$$