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# LOCAL INTEGRABILITY RESULTS IN HARMONIC ANALYSIS ON REDUCTIVE GROUPS IN LARGE POSITIVE CHARACTERISTIC

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ABSTRACT. – Let $\mathbf{G}$ be a connected reductive algebraic group over a non-Archimedean local field $\mathbb{K}$, and let $\mathfrak{g}$ be its Lie algebra. By a theorem of Harish-Chandra, if $\mathbb{K}$ has characteristic zero, the Fourier transforms of orbital integrals are represented on the set of regular elements in $\mathfrak{g}(\mathbb{K})$ by locally constant functions, which, extended by zero to all of $\mathfrak{g}(\mathbb{K})$, are locally integrable. In this paper, we prove that these functions are in fact specializations of constructible motivic exponential functions. Combining this with the Transfer Principle for integrability of [8], we obtain that Harish-Chandra's theorem holds also when $\mathbb{K}$ is a non-Archimedean local field of sufficiently large positive characteristic. Under the hypothesis that mock exponential map exists, this also implies local integrability of Harish-Chandra characters of admissible representations of $\mathbf{G}(\mathbb{K})$, where $\mathbb{K}$ is an equicharacteristic field of sufficiently large (depending on the root datum of $\mathbf{G}$) characteristic.

RÉSUMÉ. – Soit $\mathbf{G}$ un groupe algébrique réductif connexe au-dessus d'un corps local non archimédien $\mathbb{K}$, et soit $\mathfrak{g}$ son algèbre de Lie. D'après un théorème de Harish-Chandra, si $\mathbb{K}$ est de caractéristique zéro, alors les transformés de Fourier d'intégrales orbitales sont représentés, sur l'ensemble des éléments réguliers de $\mathfrak{g}(\mathbb{K})$, par des fonctions localement constantes, qui, si on les étend par zéro à tout $\mathfrak{g}(\mathbb{K})$, sont localement intégrables. Dans ce papier, nous démontrons que ces fonctions sont en fait des spécialisations de fonctions motiviques constructibles exponentielles. En combinant ceci avec le principe de transfert d'intégrabilité de [8], nous obtenons que le théorème de Harish-Chandra est valable aussi quand $\mathbb{K}$ est un corps local non archimédien de caractéristique positive suffisamment grande. Sous l'hypothèse que l'application exponentielle feinte existe, ceci implique aussi l'intégrabilité locale des caractères de Harish-Chandra de représentations admissibles de $\mathbf{G}(\mathbb{K})$, où $\mathbb{K}$ est un corps d'équicaractéristique suffisamment grande (en fonction de la donnée radicielle de $\mathbf{G}$).

## 1. Introduction

In this paper we prove an extension of Harish-Chandra's theorems about local integrability of the functions representing various distributions arising in harmonic analysis on $p$-adic groups to the positive characteristic case, when the residue characteristic is large. Our method consists in transferring Harish-Chandra's results from characteristic zero to positive characteristic. In the recent years such transfer has become a prominent technique, culminating in the transfer of the Fundamental Lemma from positive characteristic to characteristic zero, [9], [38]. Two distinct ways of carrying out transfer have been described in the literature—one method is based on the idea of close local fields, due to D. Kazhdan and J.-L. Waldspurger, cf. [39]. The other method is based on the program outlined by T.C. Hales in [22] of making harmonic analysis on reductive groups over non-Archimedean local fields "field-independent" via the use of motivic integration, and this is the method we use.

We observe that the statements we are proving in this paper are much more analytic in nature than any of the statements previously handled by the transfer methods—namely, here we talk about $L^1$-integrability, as opposed to much more algebraic-type statements about equalities between integrals of functions that are known to be integrable. In this sense it is somewhat surprising that the transfer is still possible, and it requires a new type of transfer principle, which we prove in [8]. We note that the use of this very general transfer principle allows us to avoid substantial technical difficulties that one faces when using the method of transfer based on the technique of close local fields, at the cost, however, of not getting a precise lower bound on the characteristic of the fields for which our results apply.

Our main technical result is Theorem 5.8 showing that the functions representing the Fourier transforms of the orbital integrals form a family of so-called constructible motivic exponential functions. These functions were introduced by R. Cluckers and F. Loeser in [12]; they are defined in a field-independent manner by means of logic (in fact, we use a slight generalization; see §B.3.1). Theorem 5.8 implies that Transfer principles for integrability and boundedness apply to the Fourier transforms of all orbital integrals, and in particular, to the nilpotent ones. Once all the required properties of the nilpotent orbital integrals are transferred to the positive characteristic in Theorem 2.1, the analogues of many of the classical results for general distributions follow, thanks to the work of DeBacker [14], and J. Adler and J. Korman, [3]. Thus we obtain our main results: Theorems 2.2 and 5.9 (the former assumes the hypothesis on the existence of a mock exponential map, which we review in 2.2.1).

We note that for $\mathrm{GL}_n$, the local integrability of characters was proved by Rodier [35] for $p > n$, and by a different method, by B. Lemaire [29] for arbitrary $p$. Lemaire also proved the local integrability of characters for the inner forms of $\mathrm{GL}_n$ and $\mathrm{SL}_n$, and for twisted characters of $\mathrm{GL}_n$, [30], [31].

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## 2. Results

### 2.1. Notation

For a discretely valued field $\mathbb{K}$, its ring of integers will be denoted by $\Omega_{\mathbb{K}}$, the maximal ideal by $\mathfrak{p}_{\mathbb{K}}$, and the residue field by $k_{\mathbb{K}}$.

Let $\mathcal{A}$ be the collection of all non-Archimedean local fields $\mathbb{K}$ of characteristic zero, with a chosen uniformizer $\varpi_{\mathbb{K}}$ of $\Omega_{\mathbb{K}}$, and let $\mathcal{B}$ be the collection of all local fields $\mathbb{K}$ of positive characteristic, with a uniformizer $\varpi_{\mathbb{K}}$ of $\Omega_{\mathbb{K}}$. The notation $\mathbb{K}$ will always stand for a local field that lies in $\mathcal{A} \cup \mathcal{B}$. For an integer $M > 0$, we will also often use the collections $\mathcal{A}_M$ and $\mathcal{B}_M$ of fields in $\mathcal{A}$ and $\mathcal{B}$ respectively, with residue characteristic greater than $M$.

We use Denef-Pas language $\mathcal{L}_{\mathbb{Z}}$ with coefficients in $\mathbb{Z}$—this is a first-order language of logic; roughly speaking, formulas in this language define subsets of affine spaces uniformly over all local fields $\mathbb{K} \in \mathcal{A} \cup \mathcal{B}$, (see Appendix B for precise definitions). By "definable" we shall mean, definable in the language $\mathcal{L}_{\mathbb{Z}}$. We survey all the definitions and theorems from the theory of motivic integration that we use in Appendix B. We note that if one wishes to work only with reductive groups defined over a fixed number field $E$ (with a ring of integers $\Omega$) and its completions, then one can use the language $\mathcal{L}_{\Omega}$ defined in Appendix B; all the results still apply since any language $\mathcal{L}_{\Omega}$ includes the language $\mathcal{L}_{\mathbb{Z}}$.

Throughout this paper, $\mathbf{G}$ stands for a connected reductive algebraic group over a local field $\mathbb{K}$, and $\mathfrak{g}$ for its Lie algebra. For $X \in \mathfrak{g}(\mathbb{K})$, $D_G(X)$ is the discriminant of $X$, see Appendix A for the definition.

Following Kottwitz, [27], we call a function $F(X)$, defined and locally constant on the set of regular elements $\mathfrak{g}(\mathbb{K})^{\mathrm{reg}}$, "nice" if it satisfies the following two requirements:

- when extended by zero to all of $\mathfrak{g}(\mathbb{K})$, it is locally integrable, and
- the function $|D_G(X)|^{1/2}F(X)$ is locally bounded on $\mathfrak{g}(\mathbb{K})$.

Similarly, call a function on $\mathbf{G}(\mathbb{K})$ "nice", if it satisfies the same conditions on $\mathbf{G}(\mathbb{K})$, with $D_G(X)$ replaced by its group version $D_G(g)$, namely, the coefficient at $t^r$ (where $r$ is the rank of $\mathbf{G}$) in the polynomial $\det((t+1)I - \mathrm{Ad}(g))$.

### 2.2. The statements

We refer to Appendix A for all the definitions (of orbital integrals, etc.) and a survey of the classical results.

Our main result states that the Fourier transforms of orbital integrals are represented by nice functions, in large positive characteristic.

THEOREM 2.1. – *There exists a constant $M_{\mathbf{G}}^{\mathrm{orb}} > 0$ that depends only on the absolute root datum of $\mathbf{G}$, such that for every $\mathbb{K} \in \mathcal{B}_{M_{\mathbf{G}}^{\mathrm{orb}}}$, for every $X \in \mathfrak{g}(\mathbb{K})$, the function $\widehat{\mu}_X$ is a nice function on $\mathfrak{g}(\mathbb{K})$.*

In this theorem and all similarly phrased statements below, our assertion that there exists a constant $M > 0$ that depends only on the absolute root datum of $\mathbf{G}$ such that so-and-so properties hold for $\mathbf{G}(\mathbb{K})$ with $\mathbb{K} \in \mathcal{A}_M \cup \mathcal{B}_M$, has the following meaning. As discussed in §3.1 below, given an absolute root datum $\Psi$ (which is a field-independent construct), there exist finitely many possibilities for the root data of reductive groups over non-Archimedean local fields having the absolute root datum $\Psi$. We parametrize these possibilities by points of a definable set in §3.1. Then our statement says that there exists a constant $M$ that depends only on $\Psi$, such that for every local field $\mathbb{K} \in \mathcal{A}_M \cup \mathcal{B}_M$, for all possible connected reductive groups $\mathbf{G}$ defined over $\mathbb{K}$ with absolute root datum $\Psi$, the assertions of the theorem hold.

Theorem 2.1 is proved below in §5.2.1.

Thanks to the local character expansion near a tame semisimple element, the above theorem implies that Harish-Chandra characters of admissible representations are represented by nice functions on the group, under the additional hypothesis on the existence of a so-called mock exponential map. Local character expansion in large positive characteristic is proved by DeBacker [14] near the identity, and by Adler-Korman [3] near a general tame semisimple element, if the mock exponential map exists. We start by quoting the hypothesis, which uses the notation defined in §3.3 below.

2.2.1. *The exponential map hypothesis (*[14, Hypothesis 3.2.1]*). – Suppose $r > 0$. There exists a bijective map $\mathbf{e} : \mathfrak{g}(\mathbb{K})_r \to \mathbf{G}(\mathbb{K})_r$ such that*

1. *for all pairs $x \in \mathcal{B}(\mathbf{G}, \mathbb{K})$, $s \in \mathbb{R}_{\geq r}$, we have*
   (a) $\mathbf{e}(\mathfrak{g}(\mathbb{K})_{x,s}) = \mathbf{G}(\mathbb{K})_{x,s}$,
   (b) *For all $X \in \mathfrak{g}(\mathbb{K})_{x,r}$ and for all $Y \in \mathfrak{g}(\mathbb{K})_{x,s}$, we have $\mathbf{e}(X)\mathbf{e}(Y) \equiv \mathbf{e}(X+Y) \mod \mathbf{G}(\mathbb{K})_{x,s^+}$, and*
   (c) *$\mathbf{e}$ induces a group isomorphism of $\mathfrak{g}(\mathbb{K})_{x,s}/\mathfrak{g}(\mathbb{K})_{s,s^+}$ with $\mathbf{G}(\mathbb{K})_{x,s}/\mathbf{G}(\mathbb{K})_{s,s^+}$;*
2. *for all $g \in \mathbf{G}(\mathbb{K})$ we have $\mathrm{Int}(g) \circ \mathbf{e} = \mathbf{e} \circ \mathrm{Ad}(g)$;*
3. *$\mathbf{e}$ carries $dX$ into $dg$ (where $dX$ and $dg$ are Haar measures on $\mathfrak{g}(\mathbb{K})$ and $\mathbf{G}(\mathbb{K})$, respectively, associated with the same normalization of the Haar measure on $\mathbb{K}$, cf. §3.5).*

For classical groups one can take $\mathbf{e}$ to be the Cayley transform, for all $r > 0$.

THEOREM 2.2. – *There exists a constant $M_{\mathbf{G}} > 0$ that depends only on the absolute root datum of $\mathbf{G}$, such that if $\mathbb{K} \in \mathcal{B}_{M_{\mathbf{G}}}$ and Hypothesis 2.2.1 holds for $\mathbf{G}(\mathbb{K})$ with some $r > 0$, then for every admissible representation $\pi$ of $\mathbf{G}(\mathbb{K})$, its Harish-Chandra character $\theta_\pi$ is a nice function on $\mathbf{G}(\mathbb{K})$; in particular, the integral $\int_{\mathbf{G}(\mathbb{K})} \theta_\pi(g) f(g)\, dg$ converges, and equals $\Theta_\pi(f)$, for all test functions $f \in C_c^\infty(\mathbf{G}(\mathbb{K}))$.*

We prove this theorem in §5.3.1 below.

REMARK 2.3. – DeBacker's result on the local character expansion that we use in the proof of this theorem requires, in its full strength, the assumption that Hypothesis 2.2.1 holds for $r \in \mathbb{R}$ such that $\mathfrak{g}_r = \mathfrak{g}_{\rho(\pi)^+}$, where $\rho(\pi)$ is the depth of $\pi$. Here we only use the fact that the local character expansion holds in some (definable) neighborhood of the identity, which is yielded by DeBacker's proof assuming just the existence of the mock exponential for some $r > 0$. Note also that we do not require the mock exponential map to be definable.

Finally, Theorem 2.1 also implies (thanks to a result of DeBacker) that Fourier transforms of general invariant distributions on $\mathfrak{g}(\mathbb{K})$ with support bounded modulo conjugation are represented by nice functions in a neighborhood of the origin. This is Theorem 5.9.

The rest of the main body of the paper is devoted to the proof of these theorems. Two appendices are provided for the reader's convenience—Appendix A contains a brief summary of the definitions and relevant classical results in harmonic analysis on $p$-adic groups, and