ISSN 0012-9593

*quatrième série - tome 47 fascicule 1 janvier-février 2014*

*ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE*

Thomas BARNET-LAMB & Toby GEE & David GERAGHTY & Richard TAYLOR

# *Local-global compatibility for $l = p$, II*

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

*Ann. Scient. Éc. Norm. Sup.*
4ᵉ série, t. 47, 2014, p. 165 à 179

# LOCAL-GLOBAL COMPATIBILITY FOR $l = p$, II

BY Thomas BARNET-LAMB, Toby GEE, David GERAGHTY
AND Richard TAYLOR

ABSTRACT. – We prove the compatibility at places dividing $l$ of the local and global Langlands correspondences for the $l$-adic Galois representations associated to regular algebraic essentially (conjugate) self-dual cuspidal automorphic representations of $GL_n$ over an imaginary CM or totally real field. We prove this compatibility up to semisimplification in all cases, and up to Frobenius semisimplification in the case of Shin-regular weight.

RÉSUMÉ. – Nous prouvons la compatibilité entre les correspondances de Langlands locale et globale aux places divisant $l$ pour les représentations galoisiennes $l$-adiques associées à des représentations automorphes cuspidales algébriques régulières de $\mathrm{GL}_n$ sur un corps CM ou totalement réel qui sont duales de leur conjuguée complexe à un twist près. Nous prouvons cette compatibilité à semi-simplification près dans tous les cas, et à semi-simplification de Frobenius près lorsque le poids est régulier au sens de Shin.

## Introduction.

Thanks to the work of (among others) Chenevier, Clozel, Harris, Kottwitz, Labesse, Shin and R.T., given $F$ an imaginary CM field or totally real field, and $(\Pi, \chi)$ a regular, algebraic, essentially (conjugate) self-dual automorphic representation of $\mathrm{GL}_m(\mathbb{A}_F)$, if $l$ is prime and we fix some $\imath : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$, then there is a semisimple $l$-adic Galois representation $r_{l,\imath}(\Pi) : G_F \to \mathrm{GL}_m(\overline{\mathbb{Q}}_l)$, where $G_F$ is the absolute Galois group of $F$. This representation is uniquely determined by the requirement that it satisfies local-global compatibility at the unramified places. It is also expected to satisfy local-global compatibility at all finite places; this has been established for the places not dividing $l$ by Caraiani ([10]), building on the work of Harris-Taylor, Taylor-Yoshida, Shin and Chenevier-Harris.

The second author was partially supported by NSF grant DMS-0841491, the third author was partially supported by NSF grant DMS-0635607 and the fourth author was partially supported by NSF grant DMS-0600716 and by the Oswald Veblen and Simonyi Funds at the IAS.

It is important in some applications to have this compatibility at places dividing $l$; for example, our original motivation for considering this problem was to improve the applicability of the main results of [4]; in that paper a variety of automorphy lifting theorems are proved via making highly ramified base changes, and one loses control of the level of the automorphic representations under consideration. This control can be recovered if one knows local-global compatibility at primes dividing $l$, and this is important in applications to the weight part of Serre's conjecture (cf. [2], [1]).

Our main result is as follows (see Theorem 1.1 and Corollary 1.2).

THEOREM A. – *Let $F$ be an imaginary CM field or totally real field, let $(\Pi, \chi)$ be a regular, algebraic, essentially (conjugate) self-dual automorphic representation of $\mathrm{GL}_m(\mathbb{A}_F)$ and let $\imath : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$. If $v|l$ is a place of $F$, then*

$$\imath\mathrm{WD}(r_{l,\imath}(\Pi)|_{G_{F_v}})^{\mathrm{ss}} \cong \mathrm{rec}(\Pi_v \otimes |\det|^{(1-m)/2})^{\mathrm{ss}}.$$

*Furthermore, if $\Pi$ has Shin-regular weight, then*

$$\imath\mathrm{WD}(r_{l,\imath}(\Pi)|_{G_{F_v}})^{\text{F-ss}} \cong \mathrm{rec}(\Pi_v \otimes |\det|^{(1-m)/2}).$$

Here $\mathrm{WD}(r)$ denotes the Weil-Deligne representation attached to a de Rham $l$-adic representation $r$ of the absolute Galois group of an $l$-adic field; and rec denotes the local Langlands correspondence; and F-ss denotes Frobenius semi-simplification. (See Section 1 for details on the terminology.) In fact, we prove a slight refinement of this result which gives some information about the monodromy operator in the case where $\Pi$ does not have Shin-regular weight; see Section 1 for the details of this.

The proof of Theorem A is surprisingly simple, and relies on a generalization of a base change trick that we learned from the papers [17] and [21] (see the proof of Theorem 4.3 of [17] and Section 2.2 of [21]). The idea is as follows. Suppose that $\Pi$ has Shin-regular weight. We wish to determine the Weil-Deligne representation $\imath\mathrm{WD}(r_{l,\imath}(\Pi)|_{G_{F_v}})^{\text{F-ss}}$. The monodromy may be computed after any finite base change, and in particular we may make a base change so that $\Pi$ has Iwahori-fixed vectors, which is the situation covered by [3]; so it suffices to compute the representation of the Weil group $W_{F_v}$. It is straightforward to check that in order to do so it is enough to compute the traces of the elements $\sigma \in W_{F_v}$ of nonzero valuation (that is, those elements which map to a nonzero power of the Frobenius element in the Galois group of the residue field). This trace is then computed as follows: one makes a global base change to a CM field $E/F$ such that there is a place $w$ of $E$ lying over $v$ such that $\mathrm{BC}_{E/F}(\Pi)_w$ has Iwahori-fixed vectors, and $\sigma$ is an element of $W_{E_w} \leq W_{F_v}$. By the compatibility of base change with the local Langlands correspondence, the trace of $\sigma$ on $\imath\mathrm{WD}(r_{l,\imath}(\Pi)|_{G_{F_v}})^{\text{F-ss}}$ may then be computed over $E$, where the result follows from [3].

The subtlety in this argument is that the field $E/F$ need not be Galois, so one cannot immediately appeal to solvable base change. However, it will have solvable normal closure, so that by a standard descent argument due to Harris, together with local-global compatibility for the $p$-adic Galois representations with $p \neq l$, it is enough to know that for some prime $l'$, the global Galois representation $r_{l',\imath'}(\Pi)$ is irreducible. Under the additional assumption that $\Pi$ has extremely regular weight, the existence of such an $l'$ is established in [4]. Having thus established Theorem A in the case that $\Pi$ has extremely regular and Shin-regular weight, we then pass to the general case by means of an $l$-adic interpolation argument of Chenevier and Harris, [13] and [12]. The details are in Section 3.

**Notation and terminology**

We write all matrix transposes on the left; so ${}^tA$ is the transpose of $A$. We let $B_m \subset \mathrm{GL}_m$ denote the Borel subgroup of upper triangular matrices and $T_m \subset \mathrm{GL}_m$ the diagonal torus. We let $I_m$ denote the identity matrix in $\mathrm{GL}_m$.

If $M$ is a field, we let $\overline{M}$ denote an algebraic closure of $M$ and $G_M$ the absolute Galois group $\mathrm{Gal}\,(\overline{M}/M)$. Let $\epsilon_l$ denote the $l$-adic cyclotomic character.

Let $p$ be a rational prime and $K/\mathbb{Q}_p$ a finite extension. We let $\mathcal{O}_K$ denote the ring of integers of $K$, $\wp_K$ the maximal ideal of $\mathcal{O}_K$, $k(\nu_K)$ the residue field $\mathcal{O}_K/\wp_K$, $\nu_K : K^\times \twoheadrightarrow \mathbb{Z}$ the canonical valuation and $|\ |_K : K^\times \to \mathbb{Q}^\times$ the absolute value given by $|x|_K = \#(k(\nu_K))^{-\nu_K(x)}$. We let $|\ |_K^{1/2} : K^\times \to \mathbb{R}^\times_{>0}$ denote the unique positive unramified square root of $|\ |_K$. If $K$ is clear from the context, we will sometimes write $|\ |$ for $|\ |_K$. We let $\mathrm{Frob}_K$ denote the geometric Frobenius element of $G_{k(\nu_K)}$ and $I_K$ the kernel of the natural surjection $G_K \twoheadrightarrow G_{k(\nu_K)}$. We will sometimes abbreviate $\mathrm{Frob}_{\mathbb{Q}_p}$ by $\mathrm{Frob}_p$. We let $W_K$ denote the preimage of $\mathrm{Frob}_K^{\mathbb{Z}}$ under the map $G_K \twoheadrightarrow G_{k(\nu(K))}$, endowed with a topology by decreeing that $I_K \subset W_K$ with its usual topology is an open subgroup of $W_K$. We let $\mathrm{Art}_K : K^\times \xrightarrow{\sim} W_K^{\mathrm{ab}}$ denote the local Artin map, normalized to take uniformizers to lifts of $\mathrm{Frob}_K$.

Let $\Omega$ be an algebraically closed field of characteristic 0. A Weil-Deligne representation of $W_K$ over $\Omega$ is a triple $(V, r, N)$ where $V$ is a finite dimensional vector space over $\Omega$, $r : W_K \to \mathrm{GL}(V)$ is a representation with open kernel and $N : V \to V$ is an endomorphism with $r(\sigma)Nr(\sigma)^{-1} = |\mathrm{Art}_K^{-1}(\sigma)|_K N$. We say that $(V, r, N)$ is Frobenius semisimple if $r$ is semisimple. We let $(V, r, N)^{\text{F-ss}}$ denote the Frobenius semisimplification of $(V, r, N)$ (see for instance Section 1 of [23]) and we let $(V, r, N)^{\mathrm{ss}}$ denote $(V, r^{\mathrm{ss}}, 0)$. If $\Omega$ has the same cardinality as $\mathbb{C}$, we have the notions of a Weil-Deligne representation being *pure* or *pure of weight* $k$ – see the paragraph before Lemma 1.4 of [23]. (If $N = 0$ then the representation is pure if the eigenvalues of Frobenius are Weil numbers of the same weight, but if $N$ is nonzero then the definition is more involved.)

We will let $\mathrm{rec}_K$ be the local Langlands correspondence of [16], so that if $\pi$ is an irreducible complex admissible representation of $\mathrm{GL}_n(K)$, then $\mathrm{rec}_K(\pi)$ is a Weil-Deligne representation of the Weil group $W_K$. We will write rec for $\mathrm{rec}_K$ when the choice of $K$ is clear. If $\rho$ is a continuous representation of $G_K$ over $\overline{\mathbb{Q}}_l$ with $l \neq p$ then we will write $\mathrm{WD}(\rho)$ for the corresponding Weil-Deligne representation of $W_K$. (See for instance Section 1 of [23].)

If $m \geq 1$ is an integer, we let $\mathrm{Iw}_{m,K} \subset \mathrm{GL}_m(\mathcal{O}_K)$ denote the subgroup of matrices which map to an upper triangular matrix in $\mathrm{GL}_m(k(\nu_K))$. If $\pi$ is an irreducible admissible supercuspidal representation of $\mathrm{GL}_m(K)$ and $s \geq 1$ is an integer we let $\mathrm{Sp}_s(\pi)$ be the square integrable representation of $\mathrm{GL}_{ms}(K)$ defined for instance in Section I.3 of [16]. Similarly, if $r : W_K \to \mathrm{GL}_m(\Omega)$ is an irreducible representation with open kernel and $\pi$ is the supercuspidal representation $\mathrm{rec}_K^{-1}(r)$, we let $\mathrm{Sp}_s(r) = \mathrm{rec}_K(\mathrm{Sp}_s(\pi))$. If $K'/K$ is a finite extension and if $\pi$ is an irreducible smooth representation of $\mathrm{GL}_n(K)$ we will write $\mathrm{BC}_{K'/K}(\pi)$ for the base change of $\pi$ to $K'$ which is characterized by $\mathrm{rec}_{K'}(\pi_{K'}) = \mathrm{rec}_K(\pi)|_{W_{K'}}$.

If $\rho$ is a continuous de Rham representation of $G_K$ over $\overline{\mathbb{Q}}_p$ then we will write $\mathrm{WD}(\rho)$ for the corresponding Weil-Deligne representation of $W_K$ (its construction, which is due to Fontaine, is recalled in Section 1 of [23]), and if $\tau : K \hookrightarrow \overline{\mathbb{Q}}_p$ is a continuous embedding of fields then we will write $\mathrm{HT}_\tau(\rho)$ for the multiset of Hodge-Tate numbers of $\rho$ with respect to $\tau$. Thus $\mathrm{HT}_\tau(\rho)$ is a multiset of $\dim \rho$ integers. In fact, if $W$ is a de Rham representation of $G_K$ over $\overline{\mathbb{Q}}_p$ and if $\tau : K \hookrightarrow \overline{\mathbb{Q}}_p$ then the multiset $\mathrm{HT}_\tau(W)$ contains $i$ with multiplicity $\dim_{\overline{\mathbb{Q}}_p}(W \otimes_{\tau,K} \widehat{\overline{K}}(i))^{G_K}$. Thus for example $\mathrm{HT}_\tau(\epsilon_l) = \{-1\}$.

If $F$ is a number field and $v$ a prime of $F$, we will often denote $\mathrm{Frob}_{F_v}$, $k(\nu_{F_v})$ and $\mathrm{Iw}_{m,F_v}$ by $\mathrm{Frob}_v$, $k(v)$ and $\mathrm{Iw}_{m,v}$. If $\sigma : F \hookrightarrow \overline{\mathbb{Q}}_p$ or $\mathbb{C}$ is an embedding of fields, then we will write $F_\sigma$ for the closure of the image of $\sigma$. If $F'/F$ is a soluble, finite Galois extension and if $\pi$ is a cuspidal automorphic representation of $\mathrm{GL}_m(\mathbb{A}_F)$ we will write $\mathrm{BC}_{F'/F}(\pi)$ for its base change to $F'$, an automorphic representation of $\mathrm{GL}_n(\mathbb{A}_{K'})$. If $R : G_F \to \mathrm{GL}_m(\overline{\mathbb{Q}}_l)$ is a continuous representation, we say that $R$ is *pure of weight* $w$ if for all but finitely many primes $v$ of $F$, $R$ is unramified at $v$ and every eigenvalue of $R(\mathrm{Frob}_v)$ is a Weil $(\#k(v))^w$-number. (See Section 1 of [23].) If $F$ is an imaginary CM field, we will denote its maximal totally real subfield by $F^+$ and let $c$ denote the non-trivial element of $\mathrm{Gal}\,(F/F^+)$.

## 1. Automorphic Galois representations

We recall some now-standard notation and terminology. Let $F$ be an imaginary CM field or totally real field. Let $F^+$ denote the maximal totally real subfield of $F$. By a *RAECSDC* (if $F$ is imaginary) or *RAESDC* (if $F$ is totally real) (regular, algebraic, essentially (conjugate) self dual, cuspidal) automorphic representation of $\mathrm{GL}_m(\mathbb{A}_F)$ we mean a pair $(\Pi, \chi)$ where

- $\Pi$ is a cuspidal automorphic representation of $\mathrm{GL}_m(\mathbb{A}_F)$ such that $\Pi_\infty$ has the same infinitesimal character as some irreducible algebraic representation of the restriction of scalars from $F$ to $\mathbb{Q}$ of $\mathrm{GL}_m$,
- $\chi : \mathbb{A}_{F^+}^\times/(F^+)^\times \to \mathbb{C}^\times$ is a continuous character such that $\chi_v(-1)$ is independent of $v|\infty$,
- and $\Pi^c \cong \Pi^\vee \otimes (\chi \circ \mathbf{N}_{F/F^+} \circ \det)$.

If $\chi$ is the trivial character we will often drop it from the notation and refer to $\Pi$ as a RACSDC or RASDC (regular, algebraic, (conjugate) self dual, cuspidal) automorphic representation. We will say that $(\Pi, \chi)$ has *level prime to* $l$ (resp. *level potentially prime to* $l$) if for all $v|l$ the representation $\Pi_v$ is unramified (resp. becomes unramified after a finite base change).

If $\Omega$ is an algebraically closed field of characteristic 0 we will write $(\mathbb{Z}^m)^{\mathrm{Hom}\,(F,\Omega),+}$ for the set of $a = (a_{\tau,i}) \in (\mathbb{Z}^m)^{\mathrm{Hom}\,(F,\Omega)}$ satisfying

$$a_{\tau,1} \geq \cdots \geq a_{\tau,m}.$$

Let $w \in \mathbb{Z}$. If $F$ is totally real or imaginary CM (resp. if $\Omega = \mathbb{C}$) we will write $(\mathbb{Z}^m)_w^{\mathrm{Hom}\,(F,\Omega)}$ for the subset of elements $a \in (\mathbb{Z}^m)^{\mathrm{Hom}\,(F,\Omega)}$ with

$$a_{\tau,i} + a_{\tau \circ c, m+1-i} = w$$

(resp.

$$a_{\tau,i} + a_{c \circ \tau, m+1-i} = w.)$$