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FINITENESS OF K3 SURFACES AND THE TATE CONJECTURE

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ABSTRACT. – Given a finite field k of characteristic $p \geq 5$, we show that the Tate conjecture holds for K3 surfaces over $\overline{k}$ if and only if there are only finitely many K3 surfaces defined over each finite extension of k.

RÉSUMÉ. – Étant donné un corps k fini de caractéristique $p \geq 5$, nous montrons que la conjecture de Tate pour les surfaces K3 sur $\overline{k}$ est vérifiée si et seulement s'il existe un nombre fini de surfaces K3 définies sur chaque extension finie de k.

1. Introduction

Given a class of algebraic varieties, it is reasonable to ask if there are only finitely many members defined over a given finite field. While this is clearly the case when the appropriate moduli functor is bounded, matters are often not so simple. For example, consider the case of abelian varieties of a given dimension g. There is no single moduli space parameterizing them; rather, for each integer $d \geq 1$ there is a moduli space parameterizing abelian varieties of dimension g with a polarization of degree d. It is nevertheless possible to show (see [23, Theorem 4.1], [14, Corollary 13.13]) that there are only finitely many abelian varieties over a given finite field, up to isomorphism. Another natural class of varieties where this difficulty arises is the case of K3 surfaces. As with abelian varieties, there is not a single moduli space but rather a moduli space for each even integer $d \geq 2$, parameterizing K3 surfaces with a polarization of degree d.

In this paper, we consider the finiteness question for K3 surfaces over finite fields. Given a K3 surface X defined over a finite field k of characteristic p, the Tate conjecture predicts that the natural map

$$\operatorname{Pic}(X) \otimes \mathbf{Q}_\ell \to \mathrm{H}^2_{\text{ét}}(X_{\overline{k}}, \mathbf{Q}_\ell(1))^{\mathrm{Gal}(\overline{k}/k)}$$

is surjective for $\ell \neq p$. It admits many alternate formulations; for example, it is equivalent to the statement that the Brauer group of X is finite. We say that X/k *satisfies the Tate conjecture* *over some extension k'/k (resp. $\overline{k}$)* if the Tate conjecture holds for the base change $X_{k'}$ (resp. for all base changes $X_{k'}$ with k'/k finite).

Our main result is that this conjecture is essentially equivalent to the finiteness of the set of K3 surfaces over k. Precisely:

MAIN THEOREM. – *Let k be a finite field of characteristic p.*

1. *Assume $p \geq 3$. There are only finitely many isomorphism classes of K3 surfaces over k that satisfy the Tate conjecture over $\overline{k}$.*
2. *Assume $p \geq 5$. If there are only finitely many isomorphism classes of K3 surfaces over the quadratic extension k' of k then every K3 surface over k satisfies the Tate conjecture over k'.*

In particular, if $p \geq 5$, the Tate conjecture holds for all K3 surfaces over $\overline{k}$ if and only if there are only finitely many K3 surfaces defined over each finite extension of k.

As the Tate conjecture is known for K3 surfaces of finite height in characteristic at least 5 [16], we obtain the following unconditional corollary:

COROLLARY. – *If $p \geq 5$ then there are only finitely many isomorphism classes of K3 surfaces of finite height defined over k.*

Our argument proceeds as follows. To obtain finiteness from Tate, it suffices to prove the existence of low-degree polarizations on K3 surfaces over k. In order to do this, we use the Tate conjecture in both ℓ-adic and crystalline cohomology to control the possibilities of the Néron-Severi lattice. For the other direction, we use the finiteness statement and the existence of infinitely many Brauer classes to create a K3 surface with infinitely many twisted Fourier-Mukai partners. Since this cannot happen in characteristic zero, we obtain a contradiction by proving a lifting result. This argument does not rely on [16] (as it did in an earlier version of this paper).

Notation. Throughout, k denotes a finite field of characteristic p and cardinality $q = p^f$. We fix an algebraic closure $\overline{k}$ of k.

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2. Tate implies finiteness

2.1. Discriminant bounds for the étale and crystalline lattices

In this section, we produce bounds on the discriminants of certain lattices constructed from the étale and crystalline cohomologies of K3 surfaces over k. We begin by recalling some terminology. Let A be a principal ideal domain. By a *lattice* over A, we mean a finite free A module M together with a symmetric A-linear form $(,) : M \otimes_A M \to A$. We say that M is *non-degenerate* (resp. *unimodular*) if the map $M \to \mathrm{Hom}_A(M, A)$ provided by the pairing is injective (resp. bijective). The *discriminant* of a lattice M, denoted $\mathrm{disc}(M)$, is the determinant of the matrix (e_i, e_j), where $\{e_i\}$ is a basis for M as an A-module; it is a well-defined element of $A/(A^\times)^2$. The lattice M is non-degenerate (resp. unimodular) if and only if its discriminant is non-zero (resp. a unit). Note that the valuation of $\mathrm{disc}(M)$ at a maximal ideal of A is well-defined.

We will need a simple lemma concerning discriminants:

LEMMA 2.1.1. – *Let A be a discrete valuation ring with uniformizer t. Let M be a lattice over A and let $M' \subset M$ be an A-submodule such that M/M' has length r as an A-module. Regard M' as a lattice by restricting the form from M. Then* $\mathrm{disc}(M') = t^{2r}\,\mathrm{disc}(M)$ *up to units of A.*

Proof. – Let $e_1, \ldots, e_n$ be a basis for M and let $f_1, \ldots, f_n$ be a basis for M'. Let B be the matrix (e_i, e_j) and let B' be the matrix (f_i, f_j). Thus $\mathrm{disc}(M) = \det B$ and $\mathrm{disc}(M') = \det B'$. Let $C \in M_n(A)$ be the change of basis matrix, so that $f_i = Ce_i$. Then $\det(C) = t^r$ up to units of A. As $B' = C^t BC$, the result follows. □

The following general result on discriminant bounds will be used several times in what follows.

PROPOSITION 2.1.2. – *Fix a positive integer r and a non-negative even integer w. There exist constants C and C', depending only on r, w and q, with the following property.*

Let E be a finite unramified extension of $\mathbf{Q}_\ell$ with ring of integers $\mathcal{O}$. Let M be a lattice over $\mathcal{O}$ of rank r equipped with an endomorphism ϕ. Let v_0 be the ℓ-adic valuation of $\mathrm{disc}(M)$*. Assume that the characteristic polynomial of ϕ belongs to $\mathbf{Z}[T]$, that all eigenvalues of ϕ on $M[1/\ell]$ are Weil q-integers of weight w and that $q^{w/2}$ is a semi-simple eigenvalue of ϕ on $M[1/\ell]$.*

1. *If $\ell > C$ then the discriminant $M^{\phi=q^{w/2}}$ has ℓ-adic valuation at most v_0.*
2. *The discriminant of $M^{\phi=q^{w/2}}$ has ℓ-adic valuation at most $C' + v_0$.*

Proof. – We first define the constants C and C'. Let $\mathcal{W}$ be the set of all Weil q-integers of weight w and degree at most r. It is easy to bound the coefficients of the minimal polynomial of an element of $\mathcal{W}$, and so one sees that $\mathcal{W}$ is a finite set. Let S denote the set of elements of $\mathbf{Z}[T]$ which are monic of degree r and whose roots belong to $\mathcal{W}$. Clearly, S is a finite set; enumerate its elements as $f_1(T), \ldots, f_m(T)$. We can factor each $f_i(T)$ as $g_i(T)h_i(T)$, where $g_i(T)$ is a power of $T - q^{w/2}$ and $h_i(T)$ is an element of $\mathbf{Z}[T]$ which does not have $q^{w/2}$ as a root. For each i, pick rational polynomials $a_i(T)$ and $b_i(T)$ such that

$$a_i(T)g_i(T) + b_i(T)h_i(T) = 1.$$

Let Q be the least common multiple of the denominators of the coefficients of $a_i(T)$ and $b_i(T)$. Let s be the maximal integer such that ℓ^s divides Q, for some prime ℓ, and let ℓ_0 be the largest prime dividing Q. We claim that we can take $C = \ell_0$ and $C' = 2rs$.

We now prove these claims. Thus let M and ϕ be given, and put $N = M^{\phi=q^{w/2}}$. The characteristic polynomial of ϕ belongs to S, and is thus equal to $f_i(T)$ for some i. Put $M_1 = h_i(\phi)M$ and $M_2 = g_i(\phi)M$. One easily sees that $M_1 \oplus M_2$ is a finite index $\mathcal{O}$-submodule of M and that M_1 and M_2 are orthogonal. Furthermore, M_1 is contained in N, since $q^{w/2}$ is a semi-simple eigenvalue of ϕ, and has finite index.

Suppose that $\ell > C$. Then $a_i(T)$ and $b_i(T)$ belong to $\mathcal{O}[T]$ and so $M = M_1 \oplus M_2$. Thus $\operatorname{disc}(M) = \operatorname{disc}(M_1)\operatorname{disc}(M_2)$. It follows that $\operatorname{disc}(M_1)$ has ℓ-adic valuation at most v_0. As N and M_1 are saturated in M and $M_1 \subset N$, we have $N = M_1$, and so (1) follows.

Now suppose that ℓ is arbitrary. Then $\ell^s a_i(T)$ and $\ell^s b_i(T)$ belong to $\mathcal{O}[T]$. It follows that $M_1 \oplus M_2$ contains $\ell^s M$, and so $M/(M_1 \oplus M_2)$ has length at most rs as an $\mathcal{O}$-module. Lemma 2.1.1 shows that $\operatorname{disc}(M_1)\operatorname{disc}(M_2)$ divides $\ell^{2rs}\operatorname{disc}(M)$, and thus has ℓ-adic valuation at most $2rs + v_0 = C' + v_0$. The lemma also shows that $\operatorname{disc}(N)$ divides $\operatorname{disc}(M_1)$, which proves (2). □

Let X be a K3 surface over k. For a prime number $\ell \neq p$ put

$$M_\ell(X) = \mathrm{H}^2_{\text{ét}}(X_{\overline{k}}, \mathbf{Z}_\ell), \qquad N_\ell(X) = M_\ell(X)^{\phi = q}.$$

Then $M_\ell(X)$ is a free $\mathbf{Z}_\ell$-module of rank 22, and the cup product gives it the structure of a unimodular lattice. The space $M_\ell(X)$ admits a natural $\mathbf{Z}_\ell$-linear automorphism ϕ, the geometric Frobenius element of $\operatorname{Gal}(\overline{k}/k)$. The map ϕ does not quite preserve the form, but satisfies $(\phi x, \phi y) = q^2(x, y)$. It is known [4] that the action of ϕ on $M_\ell(X)$ is semi-simple. We give $N_\ell(X)$ the structure of a lattice by restricting the form from $M_\ell(X)$.

Proposition 2.1.3. – *There exist constants $C_1 = C_1(k)$ and $C_2 = C_2(k)$ with the following properties. Let X be a K3 surface over k and let $\ell \neq p$ be a prime number. Then*

1. *For $\ell > C_1$, the discriminant of $N_\ell(X)$ has ℓ-adic valuation zero.*
2. *The discriminant of $N_\ell(X)$ has ℓ-adic valuation at most C_2.*

Proof. – This follows immediately from Proposition 2.1.2 with $r = 22$ and $w = 2$, applied to $M = M_\ell(X)$. Note that $v_0 = 0$. □

We also need a version of the above result at p. Let $W = W(k)$ be the Witt ring of k. Put

$$M_p(X) = \mathrm{H}^2_{\text{cris}}(X/W), \qquad N_p(X) = M_p(X)^{\phi_0 = p}.$$

Then $M_p(X)$ is a free W-module of rank 22, and the cup product gives it the structure of a unimodular lattice. The lattice $M_p(X)$ admits a natural semilinear automorphism ϕ_0, the crystalline Frobenius. The map $\phi = \phi_0^f$ is W-linear (where $q = p^f$). We have $(\phi_0 x, \phi_0 y) = p^2\phi_0((x, y))$. (Note: the ϕ_0 on the right is the Frobenius on W.) Since ϕ_0 is only semi-linear, $N_p(X)$ is not a W-module, but a $\mathbf{Z}_p$-module. We give $N_p(X)$ the structure of a lattice via the form on $M_p(X)$.

We say that an eigenvalue α of a linear map is semi-simple if the α-eigenspace coincides with the α-generalized eigenspace. We now come to the main result at p:

Proposition 2.1.4. – *There exists a constant $C_3 = C_3(k)$ with the following property. Let X be a K3 surface over k. Assume that q is a semi-simple eigenvalue of ϕ on $M_p(X)[1/p]$. Then the discriminant of $N_p(X)$ has p-adic valuation at most C_3.*

Proof. – Let X be given, and put $N' = M_p(X)^{\phi = q}$, so that $N_p(X) = (N')^{\phi_0 = p}$. Proposition 2.1.2, with $r = 22$ and $w = 2$, bounds the p-adic valuation of $\operatorname{disc}(N')$ (as a lattice over W) in terms of k; in fact, the produced bound is the number C_2 from the previous proposition. The following lemma (which defines a constant C_4) now shows that the p-adic valuation of $\operatorname{disc}(N_p(X))$ is bounded by $C_3 = C_2 f + 44 C_4 f$. □