ISSN 0012-9593

quatrième série - tome 47 *fascicule 2* *mars-avril 2014*

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Dawei CHEN & Martin MÖLLER

Quadratic differentials in low genus: exceptional and non-varying strata

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Ann. Scient. Éc. Norm. Sup.
4$^{\text{e}}$ série, t. 47, 2014, p. 309 à 369

QUADRATIC DIFFERENTIALS IN LOW GENUS: EXCEPTIONAL AND NON-VARYING STRATA

BY DAWEI CHEN AND MARTIN MÖLLER

ABSTRACT. – We give an algebraic way of distinguishing the components of the exceptional strata of quadratic differentials in genus three and four. The complete list of these strata is $(9, -1)$, $(6, 3, -1)$, $(3, 3, 3, -1)$ in genus three and (12), $(9, 3)$, $(6, 6)$, $(6, 3, 3)$ and $(3, 3, 3, 3)$ in genus four. The upshot of our method is a detailed study regarding the geometry of canonical curves.

This result is part of a more general investigation about the sum of Lyapunov exponents of Teichmüller curves, building on [9], [6] and [7]. Using disjointness of Teichmüller curves with divisors of Brill-Noether type on the moduli space of curves, we show that for many strata of quadratic differentials in low genus the sum of Lyapunov exponents for the Teichmüller geodesic flow is the same for all Teichmüller curves in that stratum.

RÉSUMÉ. – Nous présentons une façon algébrique de distinguer les composantes exceptionnelles des strates de l'espace de modules des différentielles quadratiques en genres trois et quatre. La liste complète de ces strates est $(9, -1)$, $(6, 3, -1)$ et $(3, 3, 3, -1)$ en genre trois, (12), $(9, 3)$, $(6, 6)$, $(6, 3, 3)$ et $(3, 3, 3, 3)$ en genre quatre, respectivement. La distinction est basée sur des propriétés géométriques du modèle canonique de ces courbes.

Ce résultat fait partie de la détermination de la somme des exposants de Lyapunov des courbes de Teichmüller, dans la continuité de [9], [6] et [7]. Pour beaucoup de strates en petit genre les courbes de Teichmüller sont disjointes des diviseurs de type Brill-Noether. On en déduit que la somme des exposants de Lyapunov de toute courbe de Teichmüller dans ces strates est égale à la somme des exposants pour la mesure à support sur toute la strate.

1. Introduction

The moduli space $\Omega\mathcal{M}_g$ of Abelian differentials, also called the Hodge bundle, parameterizes Abelian differentials ω on genus g Riemann surfaces. Let $m_1, \ldots, m_k$ be positive integers such that $\sum_{i=1}^{k} m_i = 2g - 2$. Then $\Omega\mathcal{M}_g$ decomposes into strata $\Omega\mathcal{M}_g(m_1, \ldots, m_k)$ according to the number and multiplicity of the zeros of ω. Since the Teichmüller geodesic

During the preparation of this work the first author was partially supported by NSF grant DMS-1101153 (transferred as DMS-1200329). The second author was partially supported by ERC-StG 257137.

flow preserves these strata, many problems in Teichmüller theory can be dealt with stratum by stratum.

Similarly, let $d_1, \dots, d_n$ be non-zero integers such that $\sum_{j=1}^{n} d_j = 4g-4$ and $d_j \geq -1$ for all j. The moduli space of quadratic differentials parameterizing pairs (X, q) of a genus g Riemann surface X and a quadratic differential q with at most simple poles is stratified in the same way into $\mathcal{Q}(d_1, \dots, d_n)$, namely, q has a zero of multiplicity d_i at some point p_i for $d_i > 0$ and has a simple pole at p_j for $d_j = -1$.

Not much is known on the topology of the strata. Kontsevich and Zorich determined in [14] the connected components of $\Omega\mathcal{M}_g(m_1, \dots, m_k)$. Some strata have hyperelliptic components parameterizing Abelian differentials on hyperelliptic curves that have a single zero or a pair of zeros interchangeable under the hyperelliptic involution, some strata have components distinguished by the spin parity $\dim H^0(X, \mathrm{div}(\omega)/2)$ mod 2, and the others are connected. The connected components for strata of quadratic differentials were determined by Lanneau in [16]. Some have hyperelliptic components and besides a short list of *exceptional* cases, all the other strata are connected.

To find an algebraic invariant distinguishing the exceptional cases remained an open problem. Our first main result provides a solution to this problem. Let (X, q) be a quadratic differential in $\mathcal{Q}(d_1, \dots, d_n)$. Suppose q has a zero or pole of order d_i at p_i for $1 \leq i \leq n$. Write $\mathrm{div}(q) = \sum_{i=1}^{n} d_i p_i$ as the total divisor of q and $\mathrm{div}(q)_0 = \sum_{d_i > 0} d_i p_i$ as the zero divisor of q.

THEOREM 1.1. – *Each of the strata* $(9, -1)$, $(6, 3, -1)$ *and* $(3, 3, 3, -1)$ *in genus three has precisely two connected components, distinguished by*

$$\dim H^0(X, \mathrm{div}(q)_0/3) = 1 \quad \textit{resp.} \quad \dim H^0(X, \mathrm{div}(q)_0/3) = 2.$$

We also construct the connected components using techniques from algebraic geometry. This provides a proof of the connectedness (and irreducibility) of the two components that does not rely on any geometry of flat surfaces.

For $g = 4$ we discovered that the list of exceptional strata was incomplete in [16].

THEOREM 1.2. – *Each of the strata* (12), $(9, 3)$, $(6, 6)$, $(6, 3, 3)$ *and* $(3, 3, 3, 3)$ *in genus four has precisely two non-hyperelliptic connected components, distinguished by*

$$\dim H^0(X, \mathrm{div}(q)/3) = 1 \quad \textit{resp.} \quad \dim H^0(X, \mathrm{div}(q)/3) = 2.$$

Let us describe the upshots in proving Theorems 1.1 and 1.2, see Sections 6 and 7 for details. Consider the stratum $(9, -1)$ as an example. The canonical model of a non-hyperelliptic, genus three curve X is a plane quartic. If X admits a quadratic differential q with $\mathrm{div}(q) = 9p_1 - p_2$, then there exists a unique plane cubic E such that E and X intersect at p_1 with multiplicity 9. Furthermore, we have $\mathcal{O}_E(9p_1) \sim \mathcal{O}_E(3)$, where $\mathcal{O}(1)$ is the universal line bundle of $\mathbb{P}^2$. Two possibilities can occur, either $\mathcal{O}_E(3p_1) \sim \mathcal{O}_E(1)$ or $\mathcal{O}_E(3p_1) \not\sim \mathcal{O}_E(1)$, which distinguishes the claimed two components. In order to construct these two components, we first fix E and p_1, then consider plane quartics intersecting E at p_1 with multiplicity 9, and finally quotient out the parameter space by the automorphism group of $\mathbb{P}^2$. The same idea applies to the exceptional strata in genus four, using the fact that a canonical curve of genus four is contained in a unique quadric surface in $\mathbb{P}^3$.

In order to use the parity curve E, we need to control its singularities, which boils down to a tedious local analysis. To avoid confusing the reader by technical details, we postpone the argument to Appendix B.

We remark that the criteria related to $\mathrm{div}_0(q)/3$ and $\mathrm{div}(q)/3$ are analogous to that of $\mathrm{div}(\omega)/2$ in distinguishing the odd and even spin components of certain strata of Abelian differentials, see [14] and Sections 6, 7 for more details. It is well-known that the spin parity associated to $\mathrm{div}(\omega)/2$ is a deformation invariant, but the parity associated to $\mathrm{div}_0(q)/3$ and $\mathrm{div}(q)/3$ seems only an isolated example in low genus. Indeed, one can compute $\dim H^0(X, \mathrm{div}(\omega)/2) \bmod 2$ by using the Arf invariant, see [14, Section 3]. But an interpretation of $\dim H^0(X, \mathrm{div}_0(q)/3)$ and $\dim H^0(X, \mathrm{div}(q)/3)$ in terms of flat geometry is not known. We thus leave an interesting *open question:* compute the parity of $\mathrm{div}_0(q)/3$ resp. $\mathrm{div}(q)/3$ using flat geometry only, as for the Arf invariant.

The above results were obtained in parallel with our investigation of sums of Lyapunov exponents for Teichmüller curves. In this sense, the present paper is a continuation to quadratic differentials of our paper [7]. A connected component of a stratum was called *non-varying*, if for all Teichmüller curves in this stratum the sum of Lyapunov exponents is the same, and *varying* otherwise. We proved that many strata (components) of Abelian differentials in low genus are non-varying.

Let us recall the basic idea in [7]. The Siegel-Veech area constant c, the sum of Lyapunov exponents L and the slope s determine each other for a Teichmüller curve generated by an Abelian differential in $\Omega\mathcal{M}_g(m_1, \ldots, m_k)$:

$$s = \frac{12c}{L} = 12 - \frac{12\kappa}{L},$$

where $\kappa = \frac{1}{12}\sum_{i=1}^{k} \frac{m_i(m_i+2)}{m_1+1}$, see [9] and [6]. Let $\overline{C}$ denote the closure of a Teichmüller curve C in the compactified moduli space of curves $\overline{\mathcal{M}}_g$. We want to construct a divisor D in $\overline{\mathcal{M}}_g$ such that D is disjoint with $\overline{C}$ for all Teichmüller curves C in a given stratum. In the case of Abelian differentials, $\overline{C}$ does not intersect higher boundary divisors δ_i in $\overline{\mathcal{M}}_g$ for $i > 0$. Then we can compute the slope as well as the sum of Lyapunov exponents directly from the equality $\overline{C} \cdot D = 0$.

For quadratic differentials, the hyperelliptic strata were proved to be non-varying by [9], see Corollary 2.1 for more details. Here as our second main result, we prove that many non-hyperelliptic strata of quadratic differentials in low genus are non-varying.

THEOREM 1.3. – *Consider the strata of quadratic differentials in low genus.*

(1) *In genus one, the strata* $\mathcal{Q}(n, -1^n)$ *and* $\mathcal{Q}(n-1, 1, -1^n)$ *are non-varying for* $n \geq 2$ *(Theorem 8.1).*

(2) *In genus two, there are* 12 *non-varying strata among all strata of dimension up to seven (Theorem 9.1).*

(3) *In genus three, there are* 19 *non-varying strata among all non-exceptional strata of dimension up to eight (Theorem 10.1) and* 6 *non-varying strata among all exceptional strata (Theorem 6.2).*

(4) *In genus four, there are* 8 *non-varying strata among all non-exceptional strata of dimension up to nine (Theorem 11.1) and* 7 *non-varying strata among all exceptional strata (Theorem 7.2).*

Let us explain the upshot in proving Theorem 1.3 as well as the difference from the case of Abelian differentials. For Teichmüller curves generated by quadratic differentials in $\mathcal{Q}(d_1, \ldots, d_n)$, we have a similar relation between the Siegel-Veech area constant c, the sum of (involution-invariant) Lyapunov exponents L^+ and the slope s:

$$s = \frac{12c}{L^+} = 12 - \frac{12\kappa}{L^+},$$

where $\kappa = \frac{1}{24}\sum_{j=1}^{n} \frac{d_j(d_j+4)}{d_j+2}$, see Propositions 4.1 and 4.2. By using a divisor disjoint from a Teichmüller curve $\overline{C}$, one would naturally expect to read off the value of $L^+(C)$ from the divisor class. However, in the case of quadratic differentials, Teichmüller curves may intersect higher boundary divisors, because a core curve of a cylinder may disconnect the associated flat surface for quadratic differentials, whereas this is impossible for Abelian differentials, see Remark 4.7. Thus, for a divisor D in $\overline{\mathcal{M}}_g$ with class

$$D = a\lambda + b\delta_0 + \sum c_i\delta_i,$$

even if $\overline{C} \cdot D = 0$, we cannot directly deduce the slope $s = (\overline{C} \cdot \delta)/(\overline{C} \cdot \lambda)$, where $\delta = \sum \delta_i$ is the total boundary. Therefore, for a claimed non-varying stratum of quadratic differentials, it requires a considerable amount of work using both algebraic geometry and flat geometry to study the intersection of $\overline{C}$ with higher boundary divisors δ_i occurring in the divisor class of D.

Moreover, for a number of non-varying strata we are only able to construct a disjoint divisor in the moduli space of pointed curves $\overline{\mathcal{M}}_{g,n}$, hence we lift a Teichmüller curve C to $\overline{\mathcal{M}}_{g,n}$ by marking n zeros or poles of its generating differential. Besides λ and the boundary classes, a divisor class in $\overline{\mathcal{M}}_{g,n}$ may also contain the first Chern class ω_i of the relative dualizing line bundle associated to the ith marked point. Consequently we have to understand the intersection $\overline{C} \cdot \omega_i$. This calculation is carried out in Proposition 4.2.

Among the non-varying strata in Theorem 1.3, there are three of them for which our standard method does not work. In other words, we are not able to find divisors disjoint with all Teichmüller curves in these three strata. Instead, we adapt the idea of [25] by using certain filtration of the Hodge bundle, which is treated in Appendix A.

Finally in genus five, we show that even the stratum with a unique zero is varying (Appendix C). Therefore, it seems quite plausible that our list of non-varying strata (including the known hyperelliptic strata by [9]) is complete. Nevertheless, for a varying stratum it would still be interesting to figure out the value distribution for the sums of Lyapunov exponents for all Teichmüller curves contained in the stratum.

This paper is organized as follows. In Section 2 we provide the background on strata of Abelian and quadratic differentials. A result of independent interest shows that near certain boundary strata of the moduli space the period and plumbing parameters are coordinates of strata of quadratic differentials.

In Section 3 we recall the Picard group of moduli spaces and various divisor classes. Section 4 discusses properties of Teichmüller curves generated by quadratic differentials near the boundary of the moduli space.

In order to prove disjointness of Teichmüller curves with various divisors in genus three and four along the hyperelliptic locus and the Gieseker-Petri locus, the use of the canonical