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Pierre GERMAIN & Nader MASMOUDI

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GLOBAL EXISTENCE FOR THE EULER-MAXWELL SYSTEM

BY Pierre GERMAIN AND Nader MASMOUDI

ABSTRACT. – The Euler-Maxwell system describes the evolution of a plasma when the collisions are important enough that each species is in a hydrodynamic equilibrium. In this paper we prove global existence of small solutions to this system set in the whole three-dimensional space, by combining the space-time resonance method (to obtain decay) and energy estimates (to control high frequencies). The non-integrable decay of the solutions makes it necessary to examine resonances within the energy estimate argument.

RÉSUMÉ. – Le système d'Euler-Maxwell décrit l'évolution d'un plasma quand les collisions sont suffisamment importantes pour que chaque espèce soit dans un état d'équilibre hydrodynamique. On prouve dans cet article l'existence globale de petites solutions à ce système, posé en dimension 3 d'espace, en combinant la méthode des résonances en espace-temps (pour obtenir la décroissance des solutions) et des estimations d'énergie (pour contrôler la régularité des solutions). La décroissance non intégrable des solutions impose de combiner étroitement ces deux arguments en examinant le rôle des résonances au sein des estimations d'énergie.

1. Introduction

1.1. Plasma physics and Euler-Maxwell

There are different models to describe the state of a plasma depending on several parameters such as the Debey length, the plasma frequency, the collision frequencies between the different species… Formal derivation of these models can be found in Plasma Physics textbooks (see for instance Bellan [1], Boyd and Sanderson [4], Dendy [8] and the paper [2]…).

Since the plasma consists of a very large number of interacting particles, it is appropriate to adopt a statistical approach to describe it. In the kinetic description, it is only necessary to evolve the distribution function $f_\alpha(t,x,v)$ for each species in the system. The Vlasov equation is used in this case with the Lorentz force term and a collision term. It is coupled with the Maxwell equations for the electromagnetic fields.

If collisions are important, then each species is in a local equilibrium and the plasma is treated as a fluid. More precisely it is treated as a mixture of two or more interacting fluids.

This is the two-fluid model or the so-called Euler-Maxwell system. We refer to [22] for a discussion about the possible derivation of this system from kinetic models, namely from the two species Vlasov-Boltzmann-Maxwell system. We also refer to [24, 17, 23] for more about hydrodynamic limits of the Boltzmann equation. Another level of approximation consists in treating the plasma as a single fluid either by using the fact that the mass of the electrons is much smaller than the mass of the ions or from the hydrodynamic limit which requires (in a particular limit) that the two species evolve with a common velocity and temperature [22]. This is the model which we are going to consider in this paper.

1.2. The Euler-Maxwell equation

The Cauchy problem for the one-fluid version of the Euler-Maxwell system reads

$$
\begin{cases}
\rho\left(\partial_t u + u\cdot\nabla u\right) = -\frac{p'(\rho)}{m}\nabla\rho - \frac{e\rho}{m}\left(E + \frac{1}{c}u\times B\right) \\
\partial_t\rho + \nabla\cdot(\rho u) = 0 \\
\partial_t B + c\nabla\times E = 0 \\
\partial_t E - c\nabla\times B = 4\pi e\rho u \\
\nabla\cdot E = 4\pi e(\bar\rho - \rho) \\
\nabla\cdot B = 0 \\
(u,\rho,E,B)(t=0) = (u_0,\rho_0,E_0,B_0).
\end{cases}
\tag{1.1}
$$

The unknown functions are: ρ, the density of electrons; u, the average velocity of the electrons; E, the electric field; B the magnetic field. The physical constants are: c, the speed of light; e, the charge of the electron; m, the mass of the electron. Finally, $\bar\rho$ is the uniform density of ions, and the electron gas is supposed to be barotropic, the pressure being given by $p(\rho)$.

Let us first recall a few results related to (1.1). Global existence of weak solutions was obtained for a related 1d model in [5] using compensated compactness. Also, several asymptotic problems (WKB asymptotics, incompressible limit, non-relativistic limit, quasi-neutral limit…) were studied to derive simplified models starting from the Euler-Maxwell system [33, 35, 34, 30]. We also refer to [27] where the incompressible Navier-Stokes system is studied.

Going back to our system (1.1), we notice that the last two equations above can be removed, as soon as they are satisfied at the initial time, *which we assume from now on*: they are then conserved by the flow given by the first four.

1.3. Vicinity of the trivial equilibrium state

An obvious equilibrium state of the above system is $(\rho,u,E,B) = (\bar\rho,0,0,0)$. In order to study its stability, it is instructive to linearize the above system, and compute evolution equations for its unknowns. It is convenient to introduce at this point the projections P, respectively Q onto divergence-free, respectively curl-free vector fields; they are given by

$$
Qu := \left(\frac{\nabla}{\Delta}\right)\nabla\cdot u \quad \text{and} \quad Pu := u - Qu.
$$

Split then accordingly u and E: $u = Pu + Qu$ and $E = PE + QE$. The linearized system can be written

$$\begin{cases} \left(\partial_t^2 - c_s^2\Delta + \omega_p^2\right) \begin{pmatrix} QE \\ \rho - \bar\rho \\ Qu \end{pmatrix} = 0 \\ \left(\partial_t^2 - c^2\Delta + \omega_p^2\right) \begin{pmatrix} PE \\ \nabla \times B + \frac{4\pi e \bar\rho}{c} Pu \end{pmatrix} = 0 \\ \partial_t \left(B - \frac{cm}{e} \nabla \times u\right) = 0 \end{cases}$$

where the speed of sound c_s and the plasma frequency ω_p are given by

$$c_s = \sqrt{\frac{p'(\bar\rho)}{m}} \quad \text{and} \quad \omega_p = \sqrt{\frac{4\pi e^2 \bar\rho}{m}}.$$

Thus around the equilibrium, and at a linear level, some unknowns are governed by the Klein-Gordon equation (with different speeds), whereas the quantity $B - \frac{cm}{e}\nabla \times u$ is conserved. The Klein-Gordon equations entail decay, which is one of the keys of the global stability result which we will prove; as for the quantity $B - \frac{cm}{e}\nabla \times u$, no decay is to be expected a priori. We will therefore set it to zero, which, as it turns out, is conserved by the nonlinear flow.

1.4. Adimensionalization and reductions

In the following, we set for simplicity the physical constants m, e, c, as well as $\bar\rho$ to 1. We also drop the 4π factors, since they are irrelevant. However $c_s^2 = p'(\bar\rho) = p'(1)$ remains a number less than 1. In order to simplify a little bit the estimates, we assume

$$p(\rho) \overset{\text{def}}{=} \frac{c_s^2}{3}\rho^3.$$

Finally, set

$$n \overset{\text{def}}{=} \rho - 1.$$

The Cauchy problem becomes

$$(EM) \qquad \begin{cases} \partial_t u + u \cdot \nabla u = -c_s^2 \rho \nabla \rho - E - u \times B \\ \partial_t \rho + \nabla \cdot (\rho u) = 0 \\ \partial_t B + \nabla \times E = 0 \\ \partial_t E - \nabla \times B = \rho u \\ \nabla \cdot E = -n \\ \nabla \cdot B = 0 \\ (u, n, E, B)(t = 0) = (u_0, n_0, E_0, B_0). \end{cases}$$

We shall furthermore assume that, initially,

$$B = \nabla \times u. \tag{1.2}$$

This condition is conserved by the flow of the above system: in order to see this, use the identity $u \cdot \nabla u = -u \times (\nabla \times u) + \nabla \frac{|u|^2}{2}$ to compute

$$\begin{aligned}\partial_t(B - \nabla \times u) &= \nabla \times (u \cdot \nabla u + u \times B) \\ &= \nabla \times \left(-u \times (\nabla \times u) + \nabla \frac{|u|^2}{2} \right) - \nabla \times (u \times B) \\ &= \nabla \times (u \times (B - \nabla \times u)) .\end{aligned}$$

The linearized system reads now

$$\left\{ \begin{array}{l} \left(\partial_t^2 - c_s^2 \Delta + 1\right) \begin{pmatrix} Qu \\ n \\ QE \end{pmatrix} = 0 \\ \left(\partial_t^2 - \Delta + 1\right) \begin{pmatrix} Pu \\ PE \\ B \end{pmatrix} = 0. \end{array} \right. \tag{1.3}$$

1.5. Obtained results

Prior to stating our theorem, we need to define the operator $A \overset{\text{def}}{=} \frac{\langle D \rangle}{|D|}$ (see Section 2 for the precise definition of this operator).

THEOREM 1.1. – *Assume that the resonance separation condition* (4.1) *holds; it is the case generically in* c_s. *Fix* $\alpha_0 > 0$. *Then there exist* $C_0, \varepsilon_0, N_0 > 0$ *such that: if* $\varepsilon < \varepsilon_0$, $N > N_0$ *and*

$$\left\| \langle x \rangle^{1+\alpha_0} (u_0, An_0, E_0, AB_0) \right\|_{H^N} < \varepsilon,$$

then there exists a unique global solution of (EM) *such that*

$$\sup_t \left[\langle t \rangle^{-C_0 \varepsilon} \| (u, An, E, AB)(t) \|_{H^N} + \sqrt{\langle t \rangle} \| (u, An, E, AB)(t) \|_3 \right] \lesssim \varepsilon$$

(we refer to Section 2 for the definition of the norms appearing above). Furthermore, it scatters as t goes to infinity in that there exists a solution $(u_\ell, n_\ell, E_\ell, B_\ell)$ *of the linear system* (1.3) *corresponding to initial data in* H^{N-2} *such that*

$$\| (u, n, E, B)(t) - (u_\ell, n_\ell, E_\ell, B_\ell)(t) \|_{H^{N-2}} \to 0 \quad as \quad t \to \infty.$$

REMARK 1.2. – A few observations on the hypotheses on the initial data:

– What is meant by the condition (4.1) being generic? This condition amounts to requiring that a finite number of real analytic functions of the speed of sound c_s do not vanish. The actual system seems to be too complicated to be solved analytically, but a numerical computation in [10] reveals that the condition (4.1) is met for the value $c_s = \frac{1}{5}$. Since non-zero analytic functions have separated zeros, the condition (4.1) holds except at most for a discrete set of c_s.
– The requirements on An_0 and AB_0 imply necessarily that $\int n_0 = 0$ and $\int B_0 = 0$. In particular this is consistent with the electric neutrality. Notice that this electric neutrality assumption could recently be removed for the related Euler-Poisson system, see [12].