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Frédéric BERNICOT & Sahbi KERAANI

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# ON THE GLOBAL WELL-POSEDNESS OF THE 2D EULER EQUATIONS FOR A LARGE CLASS OF YUDOVICH TYPE DATA

BY FRÉDÉRIC BERNICOT AND SAHBI KERAANI

ABSTRACT. – The study of the 2D Euler equation with non Lipschitzian velocity was initiated by Yudovich in [20] where a result of global well-posedness for essentially bounded vorticity is proved. A lot of works have been since dedicated to the extension of this result to more general spaces. To the best of our knowledge all these contributions lack the proof of at least one of the following three fundamental properties: global existence, uniqueness and regularity persistence. In this paper we introduce a Banach space containing unbounded functions for which all these properties are shown to be satisfied.

RÉSUMÉ. – L'étude de l'équation d'Euler bidimensionnelle dans un cadre non lipschitzien a été initiée par Yudovich [20], qui a montré l'existence globale pour des tourbillons initiaux bornés. Depuis, de nombreux travaux ont été dédiés à l'extension de ce résultat à des espaces plus généraux. Au meilleur de notre connaissance aucun de ces travaux ne contient de résultat où les propriétés fondamentales suivantes soient vérifiées : existence globale, unicité et propagation de la régularité. Dans cet article, nous introduisons un nouvel espace de Banach contenant des fonctions non bornées et pour lequel ces trois propriétés sont vérifiées.

## 1. Introduction

We consider the Euler system related to an incompressible inviscid fluid with constant density, namely

$$(1)\qquad \begin{cases} \partial_t u + u\cdot\nabla u + \nabla P = 0, \qquad x\in\mathbb{R}^d, t>0,\\ \nabla . u = 0,\\ u_{|t=0} = u_0. \end{cases}$$

Here, the vector field $u=(u_2,u_1,\dots,u_d)$ is a function of $(t,x)\in\mathbb{R}_+\times\mathbb{R}^d$ denoting the velocity of the fluid and the scalar function $P$ stands for the pressure. The second equation of the system $\nabla.u=0$ is the condition of incompressibility. Mathematically, it guarantees

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the preservation of Lebesgue measure by the particle-trajectory mapping (the classical flow associated to the velocity vector fields). It is worth noting that the pressure can be recovered from the velocity via an explicit Calderón-Zygmund type operator (see [5] for instance).

The question of local well-posedness of (1) with smooth data was resolved by many authors in different spaces (see for instance [5, 14]). In this context, the vorticity $\omega = \operatorname{curl} u$ plays a fundamental role. In fact, the well-known BKM criterion [2] ensures that the development of finite time singularities for these solutions is related to the blow-up of the $L^\infty$ norm of the vorticity near the maximal time existence. A direct consequence of this result is the global well-posedness of the two-dimensional Euler solutions with smooth initial data, since the vorticity satisfies the transport equation

$$\partial_t \omega + (u \cdot \nabla)\omega = 0, \tag{2}$$

and then all its $L^p$ norms are conserved.

Another class of solutions requiring lower regularity on the velocity can be considered: the weak solutions (see for instance [12, Chap 4]). They solve a weak form of the equation in the distribution sense, placing the equations in large spaces and using duality. The divergence form of Euler equations allows to put all the derivatives on the test functions and so to obtain

$$\int_0^\infty \int_{\mathbb{R}^d} (\partial_t \varphi + (u \cdot \nabla)\varphi).u \, dxdt + \int_{\mathbb{R}^d} \varphi(0,x) u_0(x) \, dx = 0,$$

for all $\varphi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^d, \mathbb{R}^d)$ with $\nabla.\varphi = 0$. In the two dimensional space and when the regularity is sufficient to give a sense to Biot-Savart law, then one can consider an alternative weak formulation: the vorticity-stream weak formulation. It consists in resolving the weak form of (2) supplemented with the Biot-Savart law:

$$u = K * \omega, \quad \text{with} \quad K(x) = \frac{x^\perp}{2\pi |x|^2}. \tag{3}$$

In this case, $(v, \omega)$ is a weak solution to the vorticity-stream formulation of the 2D Euler equation with initial data $\omega_0$ if (3) is satisfied and

$$\int_0^\infty \int_{\mathbb{R}^2} (\partial_t \varphi + u.\nabla \varphi)\omega(t,x) dxdt + \int_{\mathbb{R}^2} \varphi(0,x)\omega_0(x) dx = 0,$$

for all $\varphi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^2, \mathbb{R})$.

The questions of existence/uniqueness of weak solutions have been extensively studied and a detailed account can be found in the books [5, 14, 12]. We emphasize that, unlike the fixed-point argument, the compactness method does not guarantee the uniqueness of the solutions and then the two issues (existence/uniqueness) are usually dealt with separately. These questions have been originally addressed by Yudovich in [20] where the existence and uniqueness of weak solution to 2D Euler systems (in bounded domain) are proved under the assumptions: $u_0 \in L^2$ and $\omega_0 \in L^\infty$. Serfati [15] proved the uniqueness and existence of a solution with initial velocity and vorticity which are only bounded (without any integrability condition). There is an extensive literature on the existence of weak solution to Euler system, possibly without uniqueness, with unbounded vorticity. DiPerna-Majda [7] proved the existence of weak solution for $\omega_0 \in L^1 \cap L^p$ with $2 < p < \infty$. The $L^1$ assumption in DiPerna-Majda's paper has been removed by Giga-Miyakawa-Osada [9]. Chae [4] proved an existence result for $\omega_0$ in $L \ln^+ L$ with compact support. More recently, Taniuchi

[16] has proved the global existence (possibly without uniqueness nor regularity persistence) for $(u_0, \omega_0) \in L^\infty \times \mathrm{BMO}$. The papers [18] and [19] are concerned with the questions of existence and uniqueness of weak solutions for larger classes of vorticity. Both have intersections with the present paper and we will come back to them at the end of this section (Remark 2). A framework for measure-valued solutions can be found in [6] and [13] (see also [8] for more detailed references).

Roughly speaking, the proof of uniqueness of weak solutions requires a uniform, in time, bound of the log-Lipschitzian norm of the velocity. This "almost" Lipschitzian regularity of the velocity is enough to assure the existence and uniqueness of the associated flow (and then of the solution). Initial conditions of the type $\omega_0 \in L^\infty(\mathbb{R}^2)$ (or $\omega_0 \in \mathrm{BMO}, B^0_{\infty,\infty}, \ldots$) guarantee the log-Lipschitzian regularity of $u_0$. However, the persistence of such regularity when time varies requires an a priori bound of these quantities for the approximate-solution sequences. This is trivially done (via the conservation law) in the $L^\infty$ case but is not at all clear for the other cases. The main issue in this context is the action of Lebesgue measure preserving homeomorphisms on these spaces. In fact, it is easy to prove that all these spaces are invariant under the action of such class of homeomorphisms, but the optimal form of the constants (depending on the homeomorphisms and important for the application) are not easy to find. It is worth mentioning, in this context, that the proof by Vishik [17] of the global existence for (1) in the borderline Besov spaces is based on a refined result on the action of Lebesgue measure preserving homeomorphisms on $B^0_{\infty,1}$.

In this paper we place ourselves in some Banach space which is strictly imbricated between $L^\infty$ and BMO. Although located beyond the reach of the conservation laws of the vorticity this space has many nice properties (namely with respect to the action of the group of Lebesgue measure preserving homeomorphisms) allowing to derive the above-mentioned a priori estimates for the approximate-solution sequences.

Before going any further, let us introduce this functional space (details about BMO spaces can be found in the book of Grafakos [10]).

Definition 1. – *For a complex-valued locally integrable function on $\mathbb{R}^2$, set*

$$\|f\|_{\mathrm{LBMO}} := \|f\|_{\mathrm{BMO}} + \sup_{B_1, B_2} \frac{|\mathrm{Avg}_{B_2}(f) - \mathrm{Avg}_{B_1}(f)|}{1 + \ln\left(\frac{1-\ln r_2}{1-\ln r_1}\right)},$$

*where the supremum is taken aver all pairs of balls $B_2 = B(x_2, r_2)$ and $B_1 = B(x_1, r_1)$ in $\mathbb{R}^2$ with $0 < r_1 \le 1$ and $2B_2 \subset B_1$. Here and subsequently, we denote*

$$\mathrm{Avg}_D(g) := \frac{1}{|D|} \int_D g(x)dx,$$

*for every $g \in L^1_{loc}$ and every non negligible set $D \subset \mathbb{R}^2$. Also, for a ball $B$ and $\lambda > 0$, $\lambda B$ denotes the ball that is concentric with $B$ and whose radius is $\lambda$ times the radius of $B$.*

We recall that

$$\|f\|_{\mathrm{BMO}} := \sup_{\text{ball } B} \mathrm{Avg}_B|f - \mathrm{Avg}_B(f)|.$$

It is worth noting that if $B_2$ and $B_1$ are two balls such that $2B_2 \subset B_1$ then[(1)]

$$|\mathrm{Avg}_{B_2}(f) - \mathrm{Avg}_{B_1}(f)| \lesssim \ln(1+\frac{r_1}{r_2})\|f\|_{\mathrm{BMO}}. \tag{4}$$

In the definition of LBMO we replace the term $\ln(1+\frac{r_1}{r_2})$ by $\ln\left(\frac{1-\ln r_2}{1-\ln r_1}\right)$, which is smaller. This puts more constraints on the functions belonging to this space[(2)] and allows us to derive some crucial property on the composition of them with Lebesgue measure preserving homeomorphisms, which is the heart of our analysis.

The following statement is the main result of the paper.

Theorem 1. – *Assume $\omega_0 \in L^p \cap \mathrm{LBMO}$ with $p \in [1, 2[$. Then there exists a unique global weak solution $(v, \omega)$ to the vorticity-stream formulation of the 2D Euler equation. Besides, there exists a constant $C_0$ depending only on the $L^p \cap \mathrm{LBMO}$-norm of $\omega_0$ such that*

$$\|\omega(t)\|_{L^p\cap\mathrm{LBMO}} \leq C_0 \exp(C_0 t), \qquad \forall\, t \in \mathbb{R}_+. \tag{5}$$

Some remarks are in order.

Remark 1. – The proof gives more, namely $\omega \in \mathscr{C}(\mathbb{R}_+, L^q)$ for all $p \leq q < \infty$. Combined with the Biot-Savart law[(3)] this yields $u \in \mathscr{C}(\mathbb{R}_+, W^{1,r}) \cap \mathscr{C}(\mathbb{R}_+, L^\infty)$ for all $\frac{2p}{2-p} \leq r < \infty$.

Remark 2. – The essential point of Theorem 1 is that it provides an initial space which is strictly larger than $L^p \cap L^\infty$ (it contains unbounded elements) which is a space of existence, uniqueness and persistence of regularity at once. We emphasize that the bound (5) is crucial since it implies that $u$ is, uniformly in time, log-Lipschitzian which is the main ingredient for the uniqueness. Once this bound established the uniqueness follows from the work by Vishik [18]. In this paper Vishik also gave a result of existence (possibly without regularity persistence) in some large space characterized by growth of the partial sum of the $L^\infty$-norm of its dyadic blocs. We should also mention the result of uniqueness by Yudovich [19] which establishes uniqueness (for bounded domain) for some space which contains unbounded functions. Note also that the example of unbounded function, given in [19], belongs actually to the space LBMO (see Proposition 3 below). Our approach is different from those in [18] and [19] and uses a classical harmonic analysis “à la Stein” without appealing to the Fourier analysis (para-differential calculus).

Remark 3. – The main ingredient of the proof of (5) is a logarithmic estimate in the space $L^p \cap \mathrm{LBMO}$ (see Theorem 2 below). It would be desirable to prove this result for BMO instead of LBMO. Unfortunately, as it is proved in [3], the corresponding estimate with BMO is optimal (with the bi-Lipschitzian norm instead of the log-Lipschitzian norm of the homeomorphism) and so the argument presented here seems to be not extendable to BMO.

(1) Throughout this paper the notation $A \lesssim B$ means that there exists a positive universal constant $C$ such that $A \leq CB$.

(2) Here, we identify all functions whose difference is a constant. In Section 2, we will prove that LBMO is complete and strictly imbricated between BMO and $L^\infty$. The “$L$” in LBMO stands for “logarithmic”.

(3) If $\omega_0 \in L^p$ with $p \in [1, 2[$ then a classical Hardy-Littlewood-Sobolev inequality gives $u \in L^q$ with $\frac{1}{q} = \frac{1}{p} - \frac{1}{2}$.