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UNIFORM RECTIFIABILITY AND HARMONIC MEASURE I: UNIFORM RECTIFIABILITY IMPLIES POISSON KERNELS IN L^p

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Abstract. – We present a higher dimensional, scale-invariant version of a classical theorem of F. and M. Riesz [37]. More precisely, we establish scale invariant absolute continuity of harmonic measure with respect to surface measure, along with higher integrability of the Poisson kernel, for a domain $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, with a uniformly rectifiable boundary, which satisfies the Harnack chain condition plus an interior (but not exterior) Corkscrew condition. In a companion paper to this one [28], we also establish a converse, in which we deduce uniform rectifiability of the boundary, assuming scale invariant L^q bounds, with $q > 1$, on the Poisson kernel.

Résumé. – On présente une version invariante par échelles et en dimension supérieure à 3, d'un théorème classique de F. et M. Riesz [37]. Plus précisément, on établit l'absolue continuité de la mesure harmonique par rapport à la mesure de surface, ainsi qu'un gain d'intégrabilité pour le noyau de Poisson, pour un domaine $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, à bord uniformément rectifiable, vérifiant une condition de chaîne de Harnack et une condition de type « points d'ancrage » ou « Corkscrew » intérieure (mais pas extérieure). L'article associé [28] établit une réciproque, c'est-à-dire l'uniforme rectifiabilité du bord en supposant des estimées invariantes par échelle L^q pour $q > 1$ sur le noyau de Poisson.

1. Introduction

In [37], F. and M. Riesz showed that for a simply connected domain in the complex plane with a rectifiable boundary, harmonic measure is absolutely continuous with respect to arc length measure. A quantitative version of this theorem was obtained by Lavrentiev [35]. More generally, if only a portion of the boundary is rectifiable, Bishop and Jones [8] have shown that harmonic measure is absolutely continuous with respect to arclength on that portion.

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They also present a counter-example to show that the result of [37] may fail in the absence of some topological hypothesis (e.g., simple connectedness).

In this paper we extend the results of [37] and [35] to higher dimensions, without imposing extra assumptions on either the exterior domain or the boundary, as has been done previously. Our extension (Theorem 1.26 below) is "scale-invariant", i.e., assuming scale-invariant analogues of the hypotheses of [37], we show that harmonic measure satisfies a scale-invariant version of absolute continuity, namely the weak-A_∞ condition (cf. Definition 1.19 below). More precisely, let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be a connected, open set. We establish the weak-A_∞ property of harmonic measure, assuming that $\partial\Omega$ is uniformly rectifiable (cf. (1.13) below), and that Ω satisfies interior (but not necessarily exterior) Corkscrew and Harnack chain conditions (cf. Definitions 1.4 and 1.6 below). Uniform rectifiability is the scale-invariant version of rectifiability, while the Corkscrew and Harnack chain conditions are scale invariant analogues of the topological properties of openness and path connectedness, respectively. We emphasize that in contrast to previous work in this area in dimensions $n+1 \geq 3$, we impose no restriction on the geometry of the *exterior* domain $\Omega_{\rm ext} := \mathbb{R}^{n+1} \setminus \overline{\Omega}$, nor any extra condition on the geometry of the boundary, beyond uniform rectifiability. In particular, we do not require that any component of $\Omega_{\rm ext}$ satisfy a Corkscrew condition (as in [29], [39, 6]) or even an n-disk condition as in [17]; nor do we assume that $\partial\Omega$ contains "Big Pieces" of the boundaries of Lipschitz sub-domains of Ω, as in [7]. The absence of such assumptions is the main advance in the present paper.

In addition, in a companion paper to this one [28], written jointly with I. Uriarte-Tuero, we establish a converse, Theorem 1.28, in which we deduce uniform rectifiability of the boundary, given a certain scale invariant local L^q estimate, with $q > 1$, for the Poisson kernel (cf. (1.24)). The method of proof in [28] may be of independent interest, as it entails a novel use of "Tb" theory to obtain a free boundary result.

Taken together, the main results of the present paper and of [28], namely Theorems 1.26 and 1.28 below, may be summarized as follows (the terminology and notation used in the statement will be clarified or cross-referenced immediately afterwards):

THEOREM 1.1. – *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be a connected open set which satisfies interior Corkscrew and Harnack chain conditions, and whose boundary $\partial\Omega$ is n-dimensional Ahlfors-David regular. Then the following are equivalent:*

1. *$\partial\Omega$ is uniformly rectifiable.*
2. *For every surface ball $\Delta = \Delta(x,r) \subset \partial\Omega$, with radius $r \lesssim \operatorname{diam}\partial\Omega$, the harmonic measure $\omega^{X_\Delta} \in$ weak-$A_\infty(\Delta)$.*
3. *$\omega \ll \sigma$, and there is a $q > 1$ such that the Poisson kernel k^{X_Δ} satisfies the scale invariant L^q bound* (1.24)*, for every $\Delta = \Delta(x,r) \subset \partial\Omega$, with radius $r \lesssim \operatorname{diam}\partial\Omega$.*

REMARK 1.2. – By the counter-example of [8], one would not expect to obtain the implication (1) $\Longrightarrow$ (2), without some sort of connectivity assumption; for us, the interior Harnack chain condition plays this role.

Given a domain $\Omega \subset \mathbb{R}^{n+1}$, a "surface ball" is a set $\Delta = \Delta(x,r) := B(x,r) \cap \partial\Omega$, where $x \in \partial\Omega$, and $B(x,r)$ denotes the standard $(n+1)$-dimensional Euclidean ball of radius r centered at x. For such a surface ball Δ, we let ω^{X_Δ} denote harmonic measure for Ω, with pole at the

"Corkscrew point" X_Δ (see Definition 1.4). The Corkscrew and Harnack chain conditions, as well as the notions of Ahlfors-David regularity (ADR), uniform rectifiability (UR) and weak-A_∞, are described in Definitions 1.4, 1.6, 1.7, 1.9, and 1.19 below.

The present paper treats the direction (1) implies (2). That (2) implies (3) is well known (see the discussion following Definition 1.19). The main result in [28] is that (3) implies (1). We mention also that we obtain in the present paper an extension of (1) implies (2), in which our hypotheses are assumed to hold only in an "interior big pieces" sense (cf. Definition 1.14 and Theorem 1.27 below).

To place Theorem 1.1 in context, we review previous related work in dimension $n+1 \geq 3$. We recall that in [29], the authors introduce the notion of a "non-tangentially accessible" (NTA) domain: Ω is said to be NTA if it satisfies the Corkscrew and Harnack chain conditions ("interior Corkscrew and Harnack chain conditions"), and also if the exterior domain, $\Omega_{\text{ext}} := \mathbb{R}^{n+1} \setminus \overline{\Omega}$ (which need not be connected), satisfies the Corkscrew condition ("exterior Corkscrew condition"). The latter was relaxed in [17] to allow a sort of "weak exterior Corkscrew" condition in which the analogue of the exterior Corkscrew point is the center merely of an n-dimensional disk in Ω_{ext}, rather than of a full Euclidean ball. A key observation made in [17] was that the weak exterior Corkscrew condition is still enough to obtain local Hölder continuity at the boundary of harmonic functions which vanish on a surface ball. In [17], the authors prove that, in the presence of Ahlfors-David regularity of the boundary, the NTA condition of [29] or even its relaxed version with "weak exterior Corkscrews", implies that Ω satisfies an "interior big pieces" of Lipschitz sub-domains condition (cf. Definition 1.14 below). By a simple maximum principle argument (plus the deep result of [15]), one then almost immediately obtains a certain lower bound for harmonic measure, to wit that there are constants $\eta \in (0,1)$ and $c_0 > 0$ such that for each surface ball $\Delta \subset \partial\Omega$, and any Borel subset $A \subset \Delta$, we have

$$\omega^{X_\Delta}(A) \geq c_0, \qquad \text{whenever } \sigma(A) \geq \eta\,\sigma(\Delta). \tag{1.3}$$

In turn, still given NTA, or at least the relaxed version of [17], the latter bound self-improves to an A_∞ estimate for harmonic measure, via the comparison principle. The same A_∞ conclusion was also obtained by a different argument in [39], under the full NTA condition of [29]. In [7], the authors impose an interior Corkscrew condition, but in lieu of the Harnack chain and exterior (or weak exterior) Corkscrew conditions, the authors assume instead the consequence of these conditions deduced in [17], namely, that Ω satisfies the aforementioned condition concerning "interior big pieces" of Lipschitz sub-domains. The bound (1.3) (suitably interpreted) then holds almost immediately (again by the maximum principle), but the self-improvement argument, in the absence of the Harnack chain and exterior (or weak exterior) Corkscrew conditions, is now more problematic (indeed, the usual proofs of the comparison principle rely on Harnack's inequality and local Hölder continuity at the boundary), and the authors conclude in [7] only that ω is weak-A_∞. On the other hand, they give an example to show that this conclusion is best possible (that is, they construct a domain which satisfies the "interior big pieces" condition, but whose harmonic measure fails to be doubling). We mention also in this context the recent paper [6], in which the geometric conclusion of [17], namely the existence of "interior big pieces" of Lipschitz sub-domains, is shown to hold assuming the full NTA condition (with two-sided Corkscrews), but in which only the lower (but not the upper) bound is required in the Ahlfors-David condition (cf. (1.8)).

In the present paper, we improve the results of [7] and of [17] by removing the "big pieces of Lipschitz sub-domains" hypothesis, as well as all assumptions regarding the exterior domain. That is, in Theorem 1.26, we assume only that Ω satisfies *interior* Corkscrew and Harnack chain conditions, and that its boundary is uniformly rectifiable. More generally, in Theorem 1.27, we suppose only that these hypotheses hold in an appropriate "interior big pieces" sense (in particular, our results include those of [7] as a special case, since their Lipschitz sub-domains clearly satisfy our hypotheses). The difficulty now, and the heart of the proof, is to establish (1.3); with the latter in hand, the self-improvement to weak A_∞ proceeds as in [7]. We mention that by an unpublished example of Hrycak, UR does not, in general, imply big pieces of Lipschitz graphs(1) (that the opposite implication does hold for ADR sets is easy, and well known). Moreover, in [28] we obtain a converse which shows that the UR hypothesis is optimal. In this connection, we mention also the following observation, which was brought to our attention by M. Badger and T. Toro. Let $F \subset \mathbb{R}^2$ denote the "4 corners Cantor set" of J. Garnett (see, e.g., [19, p. 4]), and let $F^* := F \times \mathbb{R} \subset \mathbb{R}^3$ be the "cylinder" above F. Then $\Omega := \mathbb{R}^3 \setminus F^*$ satisfies the (interior) Corkscrew and Harnack chain conditions, and has a 2-dimensional ADR boundary, but the boundary is not UR, and therefore its harmonic measure is not weak-A_∞.

We conclude this historical survey by providing some additional context for our work here and in [28], namely, that our results may be viewed as a "large constant" analogue of the work of Kenig and Toro [32, 33, 34]. The latter, taken collectively, say that in the presence of a Reifenberg flatness condition and Ahlfors-David regularity, one has that $\log k \in \mathrm{VMO}$ if and only if $\nu \in \mathrm{VMO}$, where k is the Poisson kernel with pole at some fixed point, and ν is the unit normal to the boundary. Moreover, given the same background hypotheses, the condition that $\nu \in \mathrm{VMO}$ is equivalent to a uniform rectifiability (UR) condition with vanishing trace, thus $\log k \in \mathrm{VMO} \iff$ vanishing UR. On the other hand, our large constant version "almost" says "$\log k \in \mathrm{BMO} \iff UR$", given interior Corkscrews and Harnack chains. Indeed, it is well known that the A_∞ condition (i.e., weak-A_∞ plus the doubling property) implies that $\log k \in \mathrm{BMO}$, while if $\log k \in \mathrm{BMO}$ with small norm, then $k \in A_\infty$.

In order to state our results precisely, we shall first need to discuss some preliminary matters.

1.1. Notation and definitions

– We use the letters c, C to denote harmless positive constants, not necessarily the same at each occurrence, which depend only on dimension and the constants appearing in the hypotheses of the theorems (which we refer to as the "allowable parameters"). We shall also

(1) On the other hand, Azzam and Schul [5] have recently shown that every UR set contains "big pieces of big pieces of Lipschitz graphs" (see [19, pp. 15-16] or [5] for a precise formulation). This is a beautiful result, but seems inapplicable to the estimates for harmonic measure considered here: to enable essential use of the maximum principle, one would need "*interior* big pieces (cf. Definition 1.14 below) of *interior* big pieces of Lipschitz subdomains" (say, in the presence of the 1-sided NTA condition), and it is not clear that the methods of [5] would yield such a result . We do expect that the methods of the present paper could be pushed to do so, and we plan to present these arguments, with applications to more general elliptic-harmonic measures, in a forthcoming paper.