ISSN 0012-9593

quatrième série - tome 47 fascicule 4 juillet-août 2014

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

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SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Ann. Scient. Éc. Norm. Sup.
4 e série, t. 47, 2014, p. 655 à 722

THE LOCAL LANGLANDS CORRESPONDENCE FOR GL_n IN FAMILIES

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ABSTRACT. – Let k be a finite field of characteristic p, and let E be an ℓ-adic field for $\ell \neq p$. Given a representation $\rho : G_E \to \mathrm{GL}_n(A)$ where A is a local $W(k)$-algebra, we study the problem of finding an admissible $A[\mathrm{GL}_n(E)]$-module $\pi(\rho)$ that "interpolates the local Langlands correspondence" for ρ over the points of $\operatorname{Spec} A$. We formulate a precise version of this problem and show that it has at most one solution, up to isomorphism. The first author has shown [6] that when A is a Hecke algebra, and $\rho : G_{\mathbb{Q}_\ell} \to \mathrm{GL}_2(A)$ is the natural representation of $G_{\mathbb{Q}_\ell}$ over A, the corresponding $A[\mathrm{GL}_n(E)]$-module $\pi(\rho)$ exists and arises naturally in the completed cohomology of the tower of modular curves.

RÉSUMÉ. – Soit k un corps fini de caractéristique p et soit E un corps ℓ-adique où $\ell \neq p$. Étant donnée une représentation $\rho : G_E \to \mathrm{GL}_n(A)$, où A est une $W(k)$-algèbre locale, nous étudions le problème de la recherche de $A[\mathrm{GL}_n(E)]$-module admissible $\pi(\rho)$ qui « interpole la correspondance de Langlands locale » pour ρ sur les points de $\operatorname{Spec} A$. Nous formulons une version précise de ce problème et montrons qu'il a au plus une solution, à isomorphisme près. Le premier auteur a montré [6] que lorsque A est une algèbre de Hecke, et $\rho : G_{\mathbb{Q}_\ell} \to \mathrm{GL}_2(A)$ est la représentation naturelle de $G_{\mathbb{Q}_\ell}$ sur A, alors le $A[\mathrm{GL}_2(E)]$-module $\pi(\rho)$ existe et apparaît naturellement dans la cohomologie de la tour des courbes modulaires.

1. Introduction

The goal of this paper is to extend, for any non-Archimedean local field E of residue characteristic ℓ, the local Langlands correspondence between n-dimensional Weil-Deligne representations of the Weil group W_E and admissible smooth representations of $\mathrm{GL}_n(E)$ to a correspondence defined on p-adic families of representations of the absolute Galois group G_E of E (for primes p distinct from ℓ).

Fix an algebraically closed field K containing $\mathbb{Q}_p$. Then on one side of the local Langlands correspondence are the n-dimensional, Frobenius-semisimple Weil-Deligne representations of W_E; that is, pairs (ρ, N), where ρ is an n-dimensional, semisimple representation of W_E

The first author was supported in part by NSF grants DMS-0401545, DMS-0701315, and DMS-1002339.

over K, and N is a nilpotent endomorphism of the representation space of ρ such that $\rho(w)N\rho(w)^{-1} = q^{|w|}N$ for all $w \in W_E$.

On the other side are the admissible smooth K-representations of $\mathrm{GL}_n(E)$; that is, representations π of $\mathrm{GL}_n(E)$ such that every element of π is fixed by some compact open subset U of $\mathrm{GL}_n(E)$, and such that, for all such U, the space π^U of U-invariant vectors in π is finite-dimensional.

The most naive thing one could hope for would be the following: given a family of representations of G_E over a complete, Noetherian, local, p-torsion free ring A of characteristic zero and residue characteristic p (that is, a representation $\rho : G_E \to \mathrm{GL}_n(A)$ for such an A), one could hope to find an admissible smooth $A[\mathrm{GL}_n(E)]$-module π such that, for every characteristic zero prime ideal $\mathfrak{p}$ of A, with residue field $\kappa(\mathfrak{p})$, the representation $\pi \otimes_A \kappa(\mathfrak{p})$ of $\mathrm{GL}_n(E)$ corresponds, via local Langlands, with the Frobenius-semisimple Weil-Deligne representation attached to $\rho \otimes_A \kappa(\mathfrak{p})$.

A moment's consideration will show that this is far too much to hope for. Indeed, if ρ is a direct sum of two unramified characters that specializes at a single characteristic zero point $\mathfrak{p}_0$ to the trivial character plus the cyclotomic character, then the local Langlands correspondence would tell us that $\pi \otimes_A \kappa(\mathfrak{p}_0)$ would be one-dimensional. On the other hand at general points $\mathfrak{p}$ one would need $\pi \otimes_A \kappa(\mathfrak{p})$ to be infinite-dimensional.

Indeed, as this example suggests, even formulating a precise statement of a "local Langlands correspondence in families" is nontrivial. Our approach is motivated by global considerations, suggested by work of the first author [6].

1.1. Global motivation

In the setting of [6], the ring A is typically a p-adically completed Hecke algebra, and ρ is the two dimensional representation of $G_{\mathbb{Q}}$ over A arising from the theory of p-adic modular forms. For a certain finite set of primes Σ containing p, the Hecke algebra A acts on a suitable localization of p-adically completed cohomology $H^1(X_\Sigma)$ of the tower of modular curves of levels divisible by primes in Σ. This cohomology is also equipped with commuting actions of G_Q, of $\mathrm{GL}_2(\mathbb{Q}_p)$, and of $\mathrm{GL}_2(\mathbb{Q}_\ell)$ for ℓ a prime of Σ not equal to p.

What is shown in [6] is that the space $H^1(X_\Sigma)$ has a natural tensor factorization, and that the tensor factor corresponding to ℓ interpolates the local Langlands correspondence in a natural way. More precisely, one has a factorization:

$$H^1(X_\Sigma) \cong \rho \otimes \pi_p \otimes \bigotimes_{\ell \in \Sigma \setminus \{p\}} \pi_\ell$$

where π_p is a certain representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ attached to $\rho|_{G_{\mathbb{Q}_p}}$ via considerations arising from the p-adic local Langlands correspondence, and the representations π_ℓ depend only on the restriction of ρ to $G_{\mathbb{Q}_\ell}$, and interpolate the local Langlands correspondence in a natural way. (We refer the reader to Conjecture 6.1.6 of [6] for a precise statement; Proposition 6.2.13 of [6] establishes this conjecture under mild hypotheses.)

The representations π_ℓ have a number of nice properties, which suggest the "shape" that the desired "local Langlands correspondence in families" should take:

- π_ℓ is the smooth $W(k)$-dual of a (unique) admissible smooth $A[\mathrm{GL}_2(Q_\ell)]$-module π_ℓ', where k is the residue field of A. (Modules with this property are called "coadmissible" and studied in detail in Appendix C of [6].)
- π_ℓ is "A-cotorsion free" in the sense of Definition C.37 of [6].
- For a Zariski dense set of characteristic zero primes $\mathfrak{p}$ of A, the space $\pi_\ell[\mathfrak{p}]$ of $\mathfrak{p}$-torsion vectors in π_ℓ is the representation attached to $\rho \otimes_A \kappa(\mathfrak{p})$ by a "generic" version of the local Langlands correspondence due to Breuil and Schneider. (This correspondence differs from the usual local Langlands correspondence in several ways. Rather than being a bijection it is a map from isomorphism classes of n-dimensional representations of G_E to indecomposable (but not necessarily irreducible) generic representations of $\mathrm{GL}_2(E)$. We refer the reader to Section 4 for the definition and basic properties of the Breuil-Schneider correspondence.)
- The space $(\pi_\ell/p\pi_\ell)[\mathfrak{m}]$ (where $\mathfrak{m}$ is the maximal ideal of A) has an absolutely irreducible generic socle, and no other generic subquotients. (Here recall that the socle of a $k[\mathrm{GL}_n(E)]$-module is its maximal semisimple submodule; that is, the sum of all of its irreducible submodules. A generic representation of $\mathrm{GL}_n(E)$ is one whose restriction to the unipotent radical of a Borel contains a generic character; for $\mathrm{GL}_2(\mathbb{Q}_\ell)$ an irreducible admissible representation is generic if, and only if, it is infinite dimensional.) In the setting above, the genericity of the socle of $(\pi_\ell/p\pi_\ell)[\mathfrak{m}]$ is a consequence of Ihara's lemma.

Given an arbitrary A, and an arbitrary family $\rho : G_E \to \mathrm{GL}_n(A)$, it is natural in light of the above properties to ask if one can attach an $A[\mathrm{GL}_n(E)]$-module π to ρ, with properties similar to those above, and to ask if such properties characterize such a module uniquely. In fact, in order to avoid the complication of dealing with the theory of coadmissible modules, we will work in a setting dual to the above picture.

1.2. The local Langlands correspondence for GL_n in p-adic families

We are now in a position to state our main result. Before we do so, we introduce further notation: Let K be a field of characteristic zero. Then given $\rho : G_E \to \mathrm{GL}_n(K)$ as above, we write $\pi(\rho)$ for the representation attached to ρ by the generic local Langlands correspondence of Breuil-Schneider, and we write $\widetilde{\pi}(\rho)$ to denote the smooth contragredient of $\pi(\rho)$.

1.2.1. Theorem. – *Let A be a reduced complete p-torsion free Noetherian local ring with maximal ideal $\mathfrak{m}$ and finite residue field k of characteristic p. If $\rho : G_E \to \mathrm{GL}_n(A)$ is continuous (when the target is given its $\mathfrak{m}$-adic topology), then there exists at most one admissible smooth $\mathrm{GL}_n(E)$-representation V over A, up to isomorphism, satisfying the following conditions:*

1. *V is A-torsion free.*
2. *If $\mathfrak{a}$ is a minimal prime of A, with residue field $\kappa(\mathfrak{a})$, then there is a $\kappa(\mathfrak{a})$-linear $\mathrm{GL}_n(E)$-equivariant isomorphism*
$$\widetilde{\pi}\big(\kappa(\mathfrak{a}) \otimes_A \rho\big) \xrightarrow{\sim} \kappa(\mathfrak{a}) \otimes_A V.$$
3. *If we write $\overline{V} := k \otimes_A V$, then the $\mathrm{GL}_n(E)$ cosocle $\mathrm{cosoc}(\overline{V})$ of $\overline{V}$ is absolutely irreducible and generic, while the kernel of the surjection $\overline{V} \to \mathrm{cosoc}(\overline{V})$ contains no generic Jordan-Hölder factors.*

Furthermore, if such a V exists, then:

4. *There exists an open dense subset U of* $\operatorname{Spec} A[\frac{1}{p}]$*, such that for each prime $\mathfrak{p}$ in U, there is a* $\mathrm{GL}_n(E)$*-equivariant, nonzero surjection*
$$\widetilde{\pi}\big(\kappa(\mathfrak{p}) \otimes_A \rho\big) \to \kappa(\mathfrak{p}) \otimes_A V,$$
where $\kappa(\mathfrak{p})$ is the residue field of $\mathfrak{p}$. Moreover, there is an open dense subset U' of U such that if $\mathfrak{p}$ lies in U' then this surjection is an isomorphism.

The admissibility condition on V, together with properties (1) through (3) above, are (roughly) dual to the properties satisfied by the tensor factors π_ℓ. Indeed, we prove a "recognition theorem" (Theorem 6.2.15 below) that states that if the smooth $W(k)$-dual of V satisfies the properties listed for π_ℓ above (for a given ρ), then V is admissible and satisfies properties (1) through (3).

If a representation V satisfying the conditions of this theorem with respect to a given Galois representation $\rho : G_E \to \mathrm{GL}_n(A)$ exists, then we write $V := \widetilde{\pi}(\rho)$. (Note that the theorem ensures that V is unique up to isomorphism, so that $\widetilde{\pi}(\rho)$ is then uniquely determined by ρ, up to isomorphism, if it exists.) We justify this notation by thinking of representations of G_E over fields of characteristic zero as "families over a single point", so that the map $\rho \mapsto \widetilde{\pi}(\rho)$, where it is defined, extends the dual of the Breuil-Schneider correspondence from one-point families to families over local rings A as in the theorem.

Part (4) of the theorem describes the precise sense in which V interpolates the local Langlands correspondences attached to the Galois representations $\kappa(\mathfrak{p}) \otimes_A \rho$ as $\mathfrak{p}$ ranges over the points of $\operatorname{Spec} A[\frac{1}{p}]$. Conjecturally, we can take U equal to all of $\operatorname{Spec} A[\frac{1}{p}]$ in statement (4), although our results fall short of establishing this. (We refer the reader to Theorems 6.2.5 and 6.2.6 for the precise results.) On the other hand, we will give examples in Section 6 showing that the subset U' is, in general, not equal to all of $\operatorname{Spec} A[\frac{1}{p}]$.

1.2.2. Remark. – Our convention for the generic local Langlands correspondence is that the $GL_n(\mathcal{K})$-representation $\pi(\rho)$ attached to a continuous Galois representation $\rho : G_E \to \mathrm{GL}_n(\mathcal{K})$ should have generic socle. It is this convention that seems to fit best with global applications of the type considered in [6] and [7], for example. On the other hand, when working with families, it turns out to be easier to interpolate representations whose cosocle is generic. This is because one may work with admissible $A[\mathrm{GL}_n(E)]$-modules, rather than the less-familiar coadmissible modules of [6]. This explains the appearance of the various contragredient representations in Theorem 1.2.1, and in our notation for the representations that it describes.

1.3. Global applications

Although we have used the results of [6] to motivate our formulation of the local Langlands correspondence in families, in fact Theorem 1.2.1, together with the "recognition theorem" (Theorem 6.2.15) described above, are essential ingredients in the proofs of the main results of [6].

In the language of Theorem 1.2.1, the results of [6] state that (again, under mild hypotheses) when A is a suitable p-adically completed Hecke algebra, and $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_n(A)$ is the