

*quatrième série - tome 47    fascicule 4    juillet-août 2014*

*ANNALES  
SCIENTIFIQUES  
de  
L'ÉCOLE  
NORMALE  
SUPÉRIEURE*

Yonatan HARPAZ & Alexei N. SKOROBOGATOV

*Singular curves and the étale Brauer-Manin obstruction for surfaces*

---

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

# SINGULAR CURVES AND THE ÉTALE BRAUER-MANIN OBSTRUCTION FOR SURFACES

BY YONATAN HARPAZ AND ALEXEI N. SKOROBOGATOV

---

**ABSTRACT.** – We give an elementary construction of a smooth and projective surface over an arbitrary number field  $k$  that is a counterexample to the Hasse principle but has infinite étale Brauer-Manin set. Our surface has a surjective morphism to a curve with exactly one  $k$ -point such that the unique  $k$ -fibre is geometrically a union of projective lines with an adelic point and the trivial Brauer group, but no  $k$ -point.

**RÉSUMÉ.** – Nous présentons une construction élémentaire d'une surface lisse et projective sur un corps de nombres quelconque  $k$  qui constitue un contre-exemple au principe de Hasse et possède l'ensemble de Brauer-Manin infini. La surface est munie d'un morphisme surjectif vers une courbe avec un seul  $k$ -point tel que l'unique fibre rationnelle, qui géométriquement est l'union de droites projectives, a un point adélique et le groupe de Brauer trivial, mais pas de  $k$ -points.

## Introduction

For a variety  $X$  over a number field  $k$  one can study the set  $X(k)$  of  $k$ -points of  $X$  by embedding it into the topological space of adelic points  $X(\mathbb{A}_k)$ . In 1970 Manin [10] suggested to use the pairing

$$X(\mathbb{A}_k) \times \mathrm{Br}(X) \rightarrow \mathbb{Q}/\mathbb{Z}$$

provided by local class field theory. The left kernel of this pairing  $X(\mathbb{A}_k)^{\mathrm{Br}}$  is a closed subset of  $X(\mathbb{A}_k)$ , and the reciprocity law of global class field theory implies that  $X(k)$  is contained in  $X(\mathbb{A}_k)^{\mathrm{Br}}$ . The first example of a smooth and projective variety  $X$  such that  $X(k) = \emptyset$  but  $X(\mathbb{A}_k)^{\mathrm{Br}} \neq \emptyset$  was constructed in [18] (see [1] for a similar example; an earlier example conditional on the Bombieri-Lang conjecture was found in [14]). Later, Harari [6] found many varieties  $X$  such that  $X(k)$  is not dense in  $X(\mathbb{A}_k)^{\mathrm{Br}}$ . For all of these examples except for that of [14] the failure of the Hasse principle or weak approximation can be explained by the étale Brauer-Manin obstruction (introduced in [18], see also [13]): the closure of  $X(k)$  in  $X(\mathbb{A}_k)$  is contained in the étale Brauer-Manin set  $X(\mathbb{A}_k)^{\mathrm{ét}, \mathrm{Br}} \subset X(\mathbb{A}_k)^{\mathrm{Br}}$  which in these cases is smaller than  $X(\mathbb{A}_k)^{\mathrm{Br}}$ . Recently Poonen [13] constructed threefolds (fibred into

rational surfaces over a curve of genus at least 1) such that  $X(k) = \emptyset$  but  $X(\mathbb{A}_k)^{\text{ét,Br}} \neq \emptyset$ . It is known that  $X(\mathbb{A}_k)^{\text{ét,Br}}$  coincides with the set of adelic points surviving the descent obstructions defined by torsors of arbitrary linear algebraic groups (as proved in [3, 17] using [7, 19]).

In 1997 Scharaschkin and the second author independently asked the question whether  $X(k) = \emptyset$  if and only if  $X(\mathbb{A}_k)^{\text{Br}} = \emptyset$  when  $X$  is a smooth and projective curve. They also asked if the embedding of  $X(k)$  into  $X(\mathbb{A}_k)^{\text{Br}}$  defines a bijection between the closure of  $X(k)$  in  $X(\mathbb{A}_k)$  and the set of connected components of  $X(\mathbb{A}_k)^{\text{Br}}$ . Despite some evidence for these conjectures, it may be prudent to consider also their weaker analogues with  $X(\mathbb{A}_k)^{\text{ét,Br}}$  in place of  $X(\mathbb{A}_k)^{\text{Br}}$ .

In this note we give an elementary construction of a smooth and projective surface  $X$  over an arbitrary number field  $k$  that is a counterexample to the Hasse principle and has infinite étale Brauer-Manin set (Section 3). Even simpler is our counterexample to weak approximation (Section 2). This is a smooth and projective surface  $X$  over  $k$  with a unique  $k$ -point and infinite étale Brauer-Manin set  $X(\mathbb{A}_k)^{\text{ét,Br}}$ ; moreover, infinitely many elements of  $X(\mathbb{A}_k)^{\text{ét,Br}}$  have all their local components in the Zariski open set  $X \setminus X(k)$ . Following Poonen we consider families of curves parameterized by a curve with exactly one  $k$ -point. The new idea is to make the unique  $k$ -fibre a singular curve, geometrically a union of projective lines, and then use properties of rational and adelic points on singular curves.

The structure of the Picard group of a singular projective curve is well known, see [2, Section 9.2] or [9, Section 7.5]. In Section 1 we give a formula for the Brauer group of a reduced projective curve, see Theorem 1.3. A singular curve over  $k$  can have surprising properties that no smooth curve can ever have: it can contain infinitely many adelic points, only finitely many  $k$ -points or none at all, and yet have the trivial Brauer group. See Corollary 3.2 for a singular, geometrically connected, projective curve over an arbitrary number field  $k$  that is a counterexample to the Hasse principle not explained by the Brauer-Manin obstruction. In [8] the first author proves that every counterexample to the Hasse principle on a curve which geometrically is a union of projective lines, can be explained by finite descent (and hence by the étale Brauer-Manin obstruction). Here we note that geometrically connected and simply connected projective curves over number fields satisfy the Hasse principle, a statement that does not generalize to higher dimension, see Proposition 2.1 and Remark 2.2.

This paper was written while the authors were guests of the Centre interfacultaire Bernoulli of the École polytechnique fédérale de Lausanne.

## 1. The Brauer group of singular curves

Let  $k$  be a field of characteristic 0 with an algebraic closure  $\bar{k}$  and the Galois group  $\Gamma_k = \text{Gal}(\bar{k}/k)$ . For a scheme  $X$  over  $k$  we write  $\bar{X} = X \times_k \bar{k}$ . All cohomology groups in this paper are Galois or étale cohomology groups. Let  $C$  be a *reduced, geometrically connected, projective curve* over  $k$ . We define the *normalization*  $\tilde{C}$  as the disjoint union of normalizations of the irreducible components of  $C$ . The normalization morphism  $\nu : \tilde{C} \rightarrow C$  factors as

$$\tilde{C} \xrightarrow{\nu'} C' \xrightarrow{\nu''} C,$$

where  $C'$  is a maximal intermediate curve universally homeomorphic to  $C$ , see [2, Section 9.2, p. 247] or [9, Section 7.5, p. 308]. The curve  $C'$  is obtained from  $\tilde{C}$  by identifying the points which have the same image in  $C$ . In particular, there is a canonical bijection  $\nu'' : C'(K) \xrightarrow{\sim} C(K)$  for any field extension  $K/k$ . The curve  $C'$  has mildest possible singularities: for each singular point  $s \in C'(\bar{k})$  the branches of  $\tilde{C}'$  through  $s$  intersect like  $n$  coordinate axes at  $0 \in \mathbb{A}_k^n$ .

Let us define the following reduced 0-dimensional schemes:

$$(1.1) \quad \Lambda = \text{Spec}(H^0(\tilde{C}, \theta_{\tilde{C}})), \quad \Pi = C_{\text{sing}}, \quad \Psi = (\Pi \times_C \tilde{C})_{\text{red}}.$$

Here  $\Lambda$  is the  $k$ -scheme of geometric irreducible components of  $C$  (or the geometric connected components of  $\tilde{C}$ ); it is the disjoint union of closed points  $\lambda = \text{Spec}(k(\lambda))$  such that  $k(\lambda)$  is the algebraic closure of  $k$  in the function field of the corresponding irreducible component  $k(C_\lambda) = k(\tilde{C}_\lambda)$ . Next,  $\Pi$  is the union of singular points of  $C$ , and  $\Psi$  is the union of fibres of  $\nu : \tilde{C} \rightarrow C$  over the singular points of  $C$  with their reduced subscheme structure. The morphism  $\nu''$  induces an isomorphism  $(\Pi \times_C C')_{\text{red}} \xrightarrow{\sim} \Pi$ , so we can identify these schemes. Let  $i : \Pi \rightarrow C$ ,  $i' : \Pi \rightarrow C'$  and  $j : \Psi \rightarrow \tilde{C}$  be the natural closed immersions. We have a commutative diagram

$$\begin{array}{ccccc} \tilde{C} & \xrightarrow{\nu'} & C' & \xrightarrow{\nu''} & C \\ j \uparrow & & \uparrow i' & \nearrow i & \\ \Psi & \xrightarrow{\nu'} & \Pi & & \end{array}$$

The restriction of  $\nu$  to the smooth locus of  $C$  induces isomorphisms

$$\tilde{C} \setminus j(\Psi) \xrightarrow{\sim} C' \setminus i'(\Pi) \xrightarrow{\sim} C \setminus i(\Pi).$$

An algebraic group over  $\Pi$  is a product  $G = \prod_{\pi} i_{\pi*}(G_{\pi})$ , where  $\pi$  ranges over the irreducible components of  $\Pi$ ,  $i_{\pi} : \text{Spec}(k(\pi)) \rightarrow \Pi$  is the natural closed immersion, and  $G_{\pi}$  is an algebraic group over the field  $k(\pi)$ .

PROPOSITION 1.1. – (i) *The canonical maps  $\mathbb{G}_{m,C'} \rightarrow \nu'_*\mathbb{G}_{m,\tilde{C}}$  and  $\mathbb{G}_{m,C'} \rightarrow i'_*\mathbb{G}_{m,\Pi}$  give rise to the exact sequence of étale sheaves on  $C'$*

$$(1.2) \quad 0 \rightarrow \mathbb{G}_{m,C'} \rightarrow \nu'_*\mathbb{G}_{m,\tilde{C}} \oplus i'_*\mathbb{G}_{m,\Pi} \rightarrow i'_*\nu'_*\mathbb{G}_{m,\Psi} \rightarrow 0,$$

where  $\nu'_*\mathbb{G}_{m,\Psi}$  is an algebraic torus over  $\Pi$ .

(ii) *The canonical map  $\mathbb{G}_{m,C} \rightarrow \nu''*\mathbb{G}_{m,C'}$  gives rise to the exact sequence of étale sheaves on  $C$ :*

$$(1.3) \quad 0 \rightarrow \mathbb{G}_{m,C} \rightarrow \nu''*\mathbb{G}_{m,C'} \rightarrow i_*\mathcal{U} \rightarrow 0,$$

where  $\mathcal{U}$  is a commutative unipotent group over  $\Pi$ .

*Proof.* – This is essentially well known, see [2], the proofs of Propositions 9.2.9 and 9.2.10, or [9, Lemma 7.5.12]. By [11, Thm. II.2.15 (b), (c)] it is enough to check the exactness of (1.2) at each geometric point  $\bar{x}$  of  $C'$ . If  $\bar{x} \notin i'(\Pi)$ , this is obvious since locally at  $\bar{x}$  the morphism  $\nu'$  is an isomorphism, and the stalks  $(i'_*\mathbb{G}_{m,\Pi})_{\bar{x}}$  and  $(i'_*\nu'_*\mathbb{G}_{m,\Psi})_{\bar{x}}$  are zero. Now let  $\bar{x} \in i'(\Pi)$ , and let  $\theta_{\bar{x}}$  be the strict Henselisation of the local ring of  $\bar{x}$  in  $C'$ . Each geometric point  $\bar{y}$  of  $\tilde{C}$  belongs to exactly one geometric connected component of  $\tilde{C}$ , and we denote by  $\theta_{\bar{y}}$

the strict Henselisation of the local ring of  $\bar{y}$  in its geometric connected component. By the construction of  $C'$  we have an exact sequence

$$0 \longrightarrow \mathcal{O}_{\bar{x}} \longrightarrow k(\bar{x}) \times \prod_{\nu'(\bar{y})=\bar{x}} \mathcal{O}_{\bar{y}} \longrightarrow \prod_{\nu'(\bar{y})=\bar{x}} k(\bar{y}) \longrightarrow 0,$$

where  $\mathcal{O}_{\bar{y}} \rightarrow k(\bar{y})$  is the reduction modulo the maximal ideal of  $\mathcal{O}_{\bar{y}}$ , and  $k(\bar{x}) \rightarrow k(\bar{y})$  is the multiplication by  $-1$ . We obtain an exact sequence of abelian groups

$$1 \longrightarrow \mathcal{O}_{\bar{x}}^* \longrightarrow k(\bar{x})^* \times \prod_{\nu'(\bar{y})=\bar{x}} \mathcal{O}_{\bar{y}}^* \longrightarrow \prod_{\nu'(\bar{y})=\bar{x}} k(\bar{y})^* \longrightarrow 1.$$

Using [11, Cor. II.3.5 (a), (c)] one sees that this is the sequence of stalks of (1.2) at  $\bar{x}$ , so that (i) is proved.

To prove (ii) consider the exact sequence

$$0 \rightarrow \mathbb{G}_{m,C} \rightarrow \nu''_* \mathbb{G}_{m,C'} \rightarrow \nu''_* \mathbb{G}_{m,C'} / \mathbb{G}_{m,C} \rightarrow 0.$$

Since  $\nu''$  is an isomorphism away from  $i(\Pi)$ , the restriction of the sheaf  $\nu''_* \mathbb{G}_{m,C'} / \mathbb{G}_{m,C}$  to  $C \setminus i(\Pi)$  is zero, hence  $\nu''_* \mathbb{G}_{m,C'} / \mathbb{G}_{m,C} = i_* \mathcal{U}$  for some sheaf  $\mathcal{U}$  on  $\Pi$ . To see that  $\mathcal{U}$  is a unipotent group scheme it is enough to check the stalks at geometric points. Let  $\bar{x}$  be a geometric point of  $i(\Pi)$ , and let  $\bar{y}$  be the unique geometric point of  $C'$  such that  $\nu''(\bar{y}) = \bar{x}$ . Let  $\mathcal{O}_{\bar{x}}$  and  $\mathcal{O}_{\bar{y}}$  be the corresponding strictly Henselian local rings. The stalk  $(\nu''_* \mathbb{G}_{m,C'} / \mathbb{G}_{m,C})_{\bar{x}}$  is  $\mathcal{O}_{\bar{y}}^* / \mathcal{O}_{\bar{x}}^*$ , and according to [9, Lemma 7.5.12 (c)], this is a unipotent group over the field  $k(\bar{x})$ . This finishes the proof.  $\square$

REMARK 1.2. – The first part of Proposition 1.1 has an alternative proof which is easier to generalize to higher dimension. Let  $X$  be a projective  $k$ -variety with normalization morphism  $\nu : \tilde{X} \rightarrow X$ . Assume that  $\tilde{X}$ ,  $X_{\text{sing}}$  and  $\tilde{X}_{\text{crit}}$  are smooth, where  $X_{\text{sing}}$  is the singular locus of  $X$  and  $\tilde{X}_{\text{crit}} = \nu^{-1}(X_{\text{sing}}) \subseteq \tilde{X}$  is the critical locus of  $\nu$ . (This assumption is automatically satisfied when  $X$  is a curve.) The analogue of  $C'$  is the  $K$ -variety  $X'$  given by the pushout in the square

$$\begin{array}{ccc} \tilde{X}_{\text{crit}} & \xrightarrow{j} & \tilde{X} \\ g \downarrow & & \downarrow \nu' \\ X_{\text{sing}} & \xrightarrow{i'} & X'. \end{array}$$

This pushout exists in the category of  $K$ -varieties since  $i'$  is a closed embedding and  $g$  is an affine morphism of smooth projective varieties (see [4, Thm. 5.4]). One then proves that the sequence of sheaves

$$0 \longrightarrow \mathbb{G}_{m,X'} \longrightarrow \nu'_* \mathbb{G}_{m,\tilde{X}} \oplus i'_* \mathbb{G}_{m,X_{\text{sing}}} \longrightarrow \nu'_* j_* \mathbb{G}_{m,\tilde{X}_{\text{crit}}} \longrightarrow 0$$

is exact, as follows. From the definition of  $X'$  we obtain that the square

$$\begin{array}{ccc} \mathcal{O}_{X'} & \longrightarrow & \nu'_* \mathcal{O}_{\tilde{X}} \\ \downarrow & & \downarrow \\ i'_* \mathcal{O}_{X_{\text{sing}}} & \longrightarrow & \nu'_* j_* \mathcal{O}_{\tilde{X}_{\text{crit}}} \end{array}$$