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SEMI-POSITIVITY IN POSITIVE CHARACTERISTICS

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ABSTRACT. – Let $f : (X, \Delta) \to Y$ be a flat, projective family of sharply F-pure, log-canonically polarized pairs over an algebraically closed field of characteristic $p > 0$ such that $p \nmid \operatorname{ind}(K_{X/Y} + \Delta)$. We show that $K_{X/Y} + \Delta$ is nef and that $f_*(\mathscr{O}_X(m(K_{X/Y} + \Delta)))$ is a nef vector bundle for $m \gg 0$ and divisible enough. Some of the results also extend to non log-canonically polarized pairs. The main motivation of the above results is projectivity of proper subspaces of the moduli space of stable pairs in positive characteristics. Other applications are Kodaira vanishing free, algebraic proofs of corresponding positivity results in characteristic zero, and special cases of subadditivity of Kodaira-dimension in positive characteristics.

RÉSUMÉ. – Soit $f : (X, \Delta) \to Y$ une famille projective plate de paires nettement F-pures et log-canoniquement polarisées sur un corps algébriquement clos de caractéristique $p > 0$ tel que $p \nmid \operatorname{ind}(K_{X/Y} + \Delta)$. Nous montrons que $K_{X/Y} + \Delta$ est nef et que $f_*(\mathscr{O}_X(m(K_{X/Y} + \Delta)))$ est un fibré vectoriel nef pour $m \gg 0$ et qu'il est assez divisible. Certains des résultats s'étendent également aux couples non log-canoniquement polarisés. La principale motivation de ces résultats est la projectivité de sous-espaces propres de l'espace des modules des paires stables en caractéristiques positives. D'autres applications incluent des nouvelles preuves algébriques des résultats de positivité en caractéristique nulle, et un cas particulier de sous-additivité de la dimension de Kodaira de caractéristique positive.

1. Introduction

Results stating positivity of the (log-)relative canonical bundle and of the pushforwards of its powers played an important role in the development of modern algebraic geometry (e.g., [3] $\sim$ Corollary 1.9 , [11, 9, 19, 45, 21] $\sim$ Theorem 1.7, where $\sim$ denotes our statements of similar flavor). Applications are numerous: projectivity and quasi-projectivity of moduli spaces (e.g., [22, 46] $\sim$ Corollary 4.1), subadditivity of Kodaira-dimension (e.g., [45, 21] $\sim$ Corollary 4.6), Shafarevich type results about hyperbolicity of moduli spaces (e.g., [34, 1, 43]), Kodaira dimension of moduli spaces (e.g., [31, 4]), etc. Most of the proofs of the above mentioned general positivity results are either analytic or depend on Kodaira vanishing.

Either way, they work only in characteristic zero. The word "general" and "most" has to be stressed here: there are positivity results available for families of curves (e.g., [43, 22]), abelian varieties [5] and K3 surfaces [27] in positive characteristics. The aim of this article is to present positivity results available for arbitrary fiber dimensions in positive characteristics, bypassing the earlier used analytic or Kodaira vanishing type techniques. The strongest statements are in the case of (log)-canonically polarized fibers, but there are results for fibers with nef log-canonical bundles as well. As in characteristic zero, one also has to put some restrictions on singularities. Here we assume the fibers to be sharply F-pure, which corresponds to characteristic zero notion of log-canonical singularities via reduction mod p (see [42] for a survey on F-singularities, and Definition 2.4 for the definition of sharply F-pure singularities).

Some differences between our results and the characteristic zero statements mentioned above have to be stressed. First, we only claim the semi-positivity of $f_* \omega_{X/Y}^m$ for m big and divisible enough. This is a notable difference, since the characteristic zero results usually start with proving the $m = 1$ case and then deduce the rest from that. However, in positive characteristics there are known counterexamples for the semi-positivity of $f_* \omega_{X/Y}$ [30, 3.2]. So, any positivity result can hold only for $m > 1$, and its proof has to bypass the $m = 1$ case. Second, the characteristic zero results are birational in the sense that for example it is enough to assume that ω_F is big for a general fiber of F. In our results nefness of ω_F is essential, and for the semi-positivity of pushforwards we even need ω_F to be ample. Hence, our results give exactly what one needs for projectivity of moduli spaces (as in [22]), but yield subadditivity of Kodaira dimension only together with the log-Minimal Model Program in positive characteristics.

1.1. Results: normal, boundary free versions over a curve base

Here we state our results in a special, but less technical form. We assume that the spaces involved are normal and we do not add boundary divisors to our varieties. The base is also assumed to be a smooth projective curve. For the general form of the results, see Section 1.2.

We work over an algebraically closed field k of characteristic $p > 0$.

THEOREM 1.1. – *Let $f : X \to Y$ be a surjective, projective morphism from a normal variety to a smooth projective curve with normal generic fiber, such that rK_X is Cartier for some integer $r > 0$. Further assume that*

(a) *either $p \nmid r$ and the general fiber is sharply F-pure,*
(b) *or $p | r$ and the general fiber is strongly F-regular.*

Then:

(1) *If $K_{X/Y}$ is f-nef and K_{X_y} is semi-ample for generic $y \in Y$, then $K_{X/Y}$ is nef.*
(2) *If $K_{X/Y}$ is f-ample, then $f_* \mathscr{O}_X(mrK_{X/Y})$ is a nef vector bundle for $m \gg 0$.*
(3) *(A subadditivity of Kodaira dimension type corollary:) If $K_{X/Y}$ is f-semi-ample, K_{X_y} is big for generic $y \in Y$ and $g(Y) \geq 2$, then K_X is big as well.*

REMARK 1.2. – To explain the scope of the above results, let us mention a few facts about F-singularities. First, the usual singularities of the minimal model can be defined in arbitrary characteristics (e.g., [24]). Then, every S_2, G_1, sharply F-pure singularity is semi-log-canonical (i.e., the pair of its normalization and its conductor is log-canonical) and every strongly F-regular singularity is Kawamata log-terminal. Furthermore, if the (log-)canonical divisor is $\mathbb{Q}$-Cartier, then the difference between the two is "small" in both cases in a measurable sense via reductions mod p [13, 15, 44, 29, 33, 32].

For example, in dimension one sharply F-pure includes smooth and nodal singularities, and strongly F-regular includes smooth singularities. In particular, Theorem 1.1 applies to stable curves, recovering results of [43].

In dimension two, strongly F-regular singularities (without boundaries) are equivalent to Kawamata log-terminal singularities for $p > 5$ [14]. In particular Theorem 1.1 applies to stable degenerations with Kawamata log-terminal general fibers when $p > 5$, regardless of the index. Furthermore, in the sharply F-pure case, much worse singularities can be allowed in the general fibers. For example the general fiber can have nodes or a big portion of log-canonical singularities with index not divisible by p. See [14] and [28], for the actual list.

In higher dimensions one experiences similar behavior, but fewer explicitly worked out examples are known. Intuitively, the non-sharply F-pure but log canonical singularities can be thought of as being supersingular in a very strong sense. This phenomenon can be made more precise in particular cases. For example cones over abelian varieties are sharply F-pure exactly if the underlying abelian variety is ordinary.

Point (2) of Theorem 1.1 is the F-singularity version of the characteristic zero statement used to show projectivity of the moduli space of stable varieties [22, 6]. Therefore, it implies projectivity of coarse moduli spaces of certain sharply F-pure moduli functors. For the precise statement we refer the reader to Section 1.2.

Furthermore, Theorem 1.1 combined with lifting arguments gives a new algebraic proof of the following characteristic zero semi-positivity statement.

COROLLARY 1.3. – *Let $f : X \to Y$ be surjective, projective morphism from a Kawamata log terminal variety to a smooth projective curve over an algebraically closed field of characteristic zero. Let r be the index of K_X.*

(1) *If $K_{X/Y}$ is f-semi-ample, then $K_{X/Y}$ is nef.*
(2) *If $K_{X/Y}$ is f-ample, then $f_*\mathscr{O}_X(mrK_{X/Y})$ is a nef vector bundle for $m \gg 0$.*

1.2. Results: full generality

In algebraic geometry, one is frequently forced to work with pairs or even with non-normal pairs for various reasons: induction on dimension, compactification, working with non-proper varieties, etc. Hence, in the present article we put our results in a more general framework than that of Section 1.1. The actual framework that we work in is motivated by the main application, the projectivity of coarse moduli spaces, and is as follows.

NOTATION 1.4. – Let $f : X \to Y$ be a flat, relatively S_2 and G_1, equidimensional, projective morphism to a projective scheme over k and Δ a $\mathbb{Q}$-Weil divisor on X, such that

(1) Supp Δ contains neither codimension 0 points nor singular codimension 1 points of the fibers,
(2) there is a $p \nmid r > 0$, such that $r\Delta$ is a $\mathbb{Z}$-divisor, Cartier in relative codimension 1 and $\omega_{X/Y}^{[r]}(r\Delta)$ is a line bundle (note that $\omega_{X/Y}^{[r]}(r\Delta)$ is defined as $\iota_*(\omega_{U/Y}^r(r\Delta|_U))$ where $\iota : U \to X$ is the intersection of the relative Gorenstein locus and the locus where $r\Delta$ is Cartier) and
(3) for all but finitely many $y \in Y$, (X_y, Δ_y) is sharply F-pure (see Definition 2.4).

NOTATION 1.5. – Sometimes instead of the assumptions of Notation 1.4, we drop the assumption $p \nmid r$, but instead of sharply F-purity we assume strong F-regularity of (X_y, Δ_y) for all but finitely many $y \in Y$ (see [41, Definition 2.10] for the definition of strong F-regularity).

The main results of the paper are as follows.

THEOREM 1.6. – *In the situation of Notation 1.4 or Notation 1.5, if $\omega_{X/Y}^{[r]}(r\Delta)$ is f-nef and for all but finitely many $y \in Y$, $K_{X_y} + \Delta_y$ is semi-ample, then $\omega_{X/Y}^{[r]}(r\Delta)$ is nef.*

THEOREM 1.7. – *In the situation of Notation 1.4, if $\omega_{X/Y}^{[r]}(r\Delta)$ is f-ample, then $f_*(\omega_{X/Y}^{[mr]}(mr\Delta))$ is nef for all integers $m \gg 0$.*

THEOREM 1.8. – *In the situation of Notation 1.5, if $\omega_{X/Y}^{[r]}(r\Delta)$ is f-ample and Y is a smooth curve, then $f_*(\omega_{X/Y}^{[mr]}(mr\Delta))$ is nef for all integers $m \gg 0$.*

Contrary to Theorem 1.6, in Theorem 1.7 we assumed that all but finitely many fibers are sharply F-pure. In fact, when $K_{X/Y} + \Delta$ is $\mathbb{Q}$-Cartier, the locus over which the (geometric) fibers are not sharply F-pure is closed [38, Theorem B]. Hence the seemingly weaker hypothesis of Theorem 1.6 is in fact only a specialization of Notation 1.4. Further, one cannot have assumptions only on the singularities of the generic fiber, if the goal is to prove nefness of $f_*(\omega_{X/Y}^{[mr]}(mr\Delta))$. Indeed, it is easy to construct examples of families over a curve with very singular fibers (i.e., projective cones over high genus curves) for which the above sheaf is not nef. On the other hand, if only the general fiber is required to be sharply F-pure, one can still try to prove weak-positivity of $f_*\left(\omega_{X/Y}^{[mr]}(mr\Delta)\right)$. This issue is addressed in other articles (e.g., [36, 38]).

COROLLARY 1.9. – *In the situation of Notation 1.4, if $\Delta = 0$, $K_{X/Y}$ is f-ample and for every $y \in Y$, X_y is sharply F-pure, $\operatorname{Aut}(X_y)$ is finite and there are only finitely many other $y' \in Y$ such that $X_y \cong X_{y'}$, then $\det\left(f_*\omega_{X/Y}^{[m]}\right)$ is an ample line bundle for all $m \gg 0$ and divisible enough.*

The author has evidence that taking determinant can be removed from the above corollary. I.e., it can be shown that $f_*\omega_{X/Y}^{[m]}$ is ample as a vector bundle. This issue will be also addressed in upcoming articles.

In addition to the above statements, the semi-ample assumption in Theorem 1.6 can be dropped on the expense that the index r has to be 1, as stated in the following theorem.