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FOURIER-MUKAI PARTNERS OF K3 SURFACES IN POSITIVE CHARACTERISTIC

BY MAX LIEBLICH AND MARTIN OLSSON

ABSTRACT. – We study Fourier-Mukai equivalence of K3 surfaces in positive characteristic and show that the classical results over the complex numbers all generalize. The key result is a positive-characteristic version of the Torelli theorem that uses the derived category in place of the Hodge structure on singular cohomology; this is proven by algebraizing formal lifts of Fourier-Mukai kernels to characteristic zero. As a consequence, any Shioda-supersingular K3 surface is uniquely determined up to isomorphism by its derived category of coherent sheaves. We also study different realizations of Mukai's Hodge structure in algebraic cohomology theories (étale, crystalline, de Rham) and use these to prove: 1) the zeta function of a K3 surface is a derived invariant (discovered independently by Huybrechts); 2) the variational crystalline Hodge conjecture holds for correspondences arising from Fourier-Mukai kernels on products of two K3 surfaces.

RÉSUMÉ. – Nous étudions les équivalences de Fourier-Mukai entre surfaces de type K3 en caractéristique positive et démontrons que les résultats classiques sur les complexes se généralisent sans modifications. Le résultat clef est un "théorème de Torelli" pour les catégories dérivées. Comme conséquence, toute K3 surface supersingulière est determinée uniquement à isomorphisme près par sa catégorie dérivée. Nous étudions de plus quelques réalisations algébriques de structure de Mukai-Hodge et les utilisons pour prouver que : 1) la fonction zêta d'une surface de type K3 est une invariante dérivée (découverte indépendamment par Huybrechts) ; 2) la conjecture variationnelle cristalline de Hodge est vérifiée pour les correspondances entre produits de surfaces de type K3 résultant de transformés de Fourier-Mukai.

1. Introduction

In this paper we establish several basic facts about Fourier-Mukai equivalence of K3 surfaces over fields of positive characteristic and develop some foundational material on deformation and lifting of Fourier-Mukai kernels, including the study of several "realizations" of Mukai's Hodge structure in standard cohomology theories (étale, crystalline, Chow, etc.).

In particular, we prove the following theorem, extending to positive characteristic classical results due to Hosono et al. [10], Mukai [25], and Orlov [29] in characteristic 0. For a scheme Z of finite type over a field k, let $\mathrm{D}(Z)$ denote the bounded derived category with coherent cohomology. For a K3 surface X over an algebraically closed field k, we have algebraic moduli spaces $\mathrm{M}_X(v)$ of sheaves with fixed Mukai vector v (see Section 3.15 for the precise definition) that are stable with respect to a suitable polarization.

THEOREM 1.1. – *Let X be a* K3 *surface over an algebraically closed field k of positive characteristic $\neq 2$.*

1. *If Y is a smooth projective k-scheme such that there exists an equivalence of triangulated categories $\mathrm{D}(X) \simeq \mathrm{D}(Y)$, then Y is a* K3 *surface isomorphic to $\mathrm{M}_X(v)$ for some Mukai vector v such that there exists a Mukai vector w with $\langle v, w\rangle = 1$.*
2. *There exist only finitely many smooth projective k-schemes Y with $\mathrm{D}(X) \simeq \mathrm{D}(Y)$.*
3. *If X has Picard number at least 11 and Y is a smooth projective k-scheme with $\mathrm{D}(Y) \simeq \mathrm{D}(X)$, then $X \simeq Y$. In particular, any Shioda-supersingular* K3 *surface is determined up to isomorphism by its derived category.*

The classical proofs of these results in characteristic 0 rely heavily on the Torelli theorem and lattice theory, so a transposition into characteristic p is necessarily delicate. We present here a theory of the "Mukai motive", generalizing the Mukai-Hodge structure to other cohomology theories, and use various realizations to aid in lifting derived-equivalence problems to characteristic 0.

These techniques also yield proofs of several other results. The first answers a question of Mustaţă and Huybrechts, while the second establishes the truth of the variational crystalline Hodge conjecture [24, Conjecture 9.2] in some special cases. (In the course of preparing this manuscript, we learned that Huybrechts discovered essentially the same proof of Theorem 1.2, in ℓ-adic form.)

THEOREM 1.2. – *If X and Y are* K3 *surfaces over a finite field* $\mathbf{F}$ *of characteristic $\neq 2$ such that $\mathrm{D}(X)$ is equivalent to $\mathrm{D}(Y)$, then X and Y have the same zeta-function. In particular, $\#X(\mathbf{F}) = \#Y(\mathbf{F})$.*

THEOREM 1.3. – *Suppose X and Y are* K3 *surfaces over an algebraically closed field k of positive characteristic $\neq 2$ with Witt vectors W, and that $\mathcal{X}/W$ and $\mathcal{Y}/W$ are lifts, giving rise to a Hodge filtration on the F-isocrystal $H^4_{\mathrm{cris}}(X \times Y/K)$, where K denotes the field of fractions of W. Suppose $Z \subset X \times Y$ is a correspondence coming from a Fourier-Mukai kernel. If the fundamental class of Z lies in $\mathrm{Fil}^2\, H^4_{\mathrm{cris}}(X \times Y/K)$ then Z is the specialization of a cycle on $\mathcal{X} \times_W \mathcal{Y}$.*

Throughout this paper we consider only fields of characteristic $\neq 2$.

1.4. Outline of the paper

Sections 2 and 3 contain foundational background material on Fourier-Mukai equivalences. In Section 2 we discuss variants in other cohomology theories (étale, crystalline, Chow) of Mukai's original construction of a Hodge structure associated to a smooth even dimensional proper scheme. In Section 3 we discuss various basic material on kernels of Fourier-Mukai equivalences. The main technical tool is Proposition 3.3, which will be used when deforming kernels. The results of these two sections are presumably well-known to experts.

As an application of the formalism of Mukai motives we prove Theorem 1.2 in Section 4.

In Section 5 we discuss the relationship between moduli of complexes and Fourier-Mukai kernels. This relationship is the key to the deformation theory arguments that follow and appears never to have been written down in this way. The main result of this section is Corollary 5.5.

Section 6 contains the key result for the whole paper (Theorem 6.1). This result should be viewed as a derived category version of the classical Torelli theorem for K3 surfaces. It appears likely that this kind of reduction to the universal case via moduli stacks of complexes should be useful in other contexts.

Using these deformation theory techniques we prove Theorem 1.3 in Section 7 and statement (1) in Theorem 1.1 in Section 8.

In Section 9 we prove statement (2) in Theorem 1.1. Our proof involves deforming to characteristic 0, which in particular is delicate for supersingular K3 surfaces.

Finally there is an appendix containing a technical result about versal deformations of polarized K3 surfaces which is used in Section 7. The main result of the appendix is Theorem A.1 concerning the Picard group of the general deformation of a fixed K3 surface from characteristic p to characteristic 0.

1.5. Notation

For a proper smooth scheme X over a field k we write $K(X)$ for the Grothendieck group of vector bundles on X and $A^*(X)_{\mathbf{Q}}$ for the Chow ring of algebraic cycles on X modulo rational equivalence. We write $\mathrm{ch} : K(X) \to A^*(X)_{\mathbf{Q}}$ for the Chern character.

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2. Mukai motive

2.1. Mukai's original construction over C: the Hodge realization [25]

Suppose X is a smooth projective variety of even dimension $d = 2\delta$. The singular cohomology $\mathrm{H}^i(X, \mathbf{Z})$ carries a natural pure Hodge structure of weight i, and the cup product defines a pairing of Hodge structures

$$\mathrm{H}^i(X, \mathbf{Z}) \times \mathrm{H}^{2d-i}(X, \mathbf{Z}) \to \mathrm{H}^{2d}(X, \mathbf{Z}) = \mathbf{Z}(-d),$$

where $\mathbf{Z}(-1)$ is the usual Tate Hodge structure of weight 2.