ISSN 0012-9593

quatrième série - tome 48 *fascicule 5* *septembre-octobre 2015*

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

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SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

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Édition / *Publication*

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Tarifs

Europe : 515 €. Hors Europe : 545 €. Vente au numéro : 77 €.

ISSN 0012-9593

Directeur de la publication : Marc Peigné
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4^e série, t. 48, 2015, p. 1075 à 1112

AMBITORIC GEOMETRY II: EXTREMAL TORIC SURFACES AND EINSTEIN 4-ORBIFOLDS

BY VESTISLAV APOSTOLOV, DAVID M. J. CALDERBANK AND PAUL GAUDUCHON

ABSTRACT. – We provide an explicit resolution of the existence problem for extremal Kähler metrics on toric 4-orbifolds M with second Betti number $b_2(M) = 2$. More precisely we show that M admits such a metric if and only if its rational Delzant polytope (which is a labelled quadrilateral) is K-polystable in the relative, toric sense (as studied by S. Donaldson, E. Legendre, G. Székelyhidi et al.). Furthermore, in this case, the extremal Kähler metric is *ambitoric*, i.e., compatible with a conformally equivalent, oppositely oriented toric Kähler metric, which turns out to be extremal as well. These results provide a computational test for the K-stability of labelled quadrilaterals.

Extremal ambitoric structures were classified locally in Part I of this work, but herein we only use the straightforward fact that explicit Kähler metrics obtained there are extremal, and the identification of Bach-flat (conformally Einstein) examples among them. Using our global results, the latter yield countably infinite families of compact toric Bach-flat Kähler orbifolds, including examples which are globally conformally Einstein, and examples which are conformal to complete smooth Einstein metrics on an open subset, thus extending the work of many authors.

RÉSUMÉ. – Nous donnons une solution complète et explicite du problème d'existence de métriques kählériennes extrémales sur un orbifold torique M de dimension réelle 4, dont le nombre de Betti $b_2(M)$ est égal à 2. Nous montrons plus précisément que M admet de telles métriques si et seulement si son polytope de Delzant rationnel — qui est alors un quadrilatère étiqueté — est K-polystable, suivant la théorie générale développée dans le cas torique par S. K. Donaldson, E. Legendre, G. Székelyhidi et al., et que ces métriques sont alors ambitoriques, donc complètement explicites d'après la classification figurant dans la première partie de ce travail. Notre approche donne de surcroît une façon effective de tester la stabilité des quadrilatères étiquetés. Parmi les métriques kählériennes construites dans cet article figurent celles dont le tenseur de Bach est nul, qui sont à la fois extrémales et conformément Einstein. Nous obtenons ainsi, en dimension 4, de nouveaux exemples explicites d'orbifolds d'Einstein compacts ou de variétés d'Einstein non-compactes, complètes et lisses.

Introduction

This paper concerns the explicit construction of extremal Kähler metrics on compact 4-orbifolds, including Kähler metrics which are conformally Einstein (either globally or on the complement of real hypersurface). The examples we construct are toric with second Betti number two, i.e., their rational Delzant polytope (which is the image of the momentum map of the 2-torus action [22, 45]) is a quadrilateral. More precisely, we use extremal *ambitoric* metrics, which we classified locally in Part I of this work, to resolve completely the existence problem in the quadrilateral case.

There are several narratives to which this paper may be viewed as a contribution. A general theme is the interplay between the abstract existence theory for a geometric PDE, and the construction of explicit solutions associated to special geometric structures. Extremal Kähler metrics were introduced by E. Calabi [16, 17] to address the problem of finding canonical Kähler metrics with Kähler form in a given cohomology class Ω on a compact complex manifold. The L_2 norm of the scalar curvature yields a functional on Ω, and its critical points are the extremal metrics. They are thus natural generalizations of constant curvature metrics on Riemann surfaces; in general, the Euler-Lagrange equation asserts that a Kähler metric is extremal if its scalar curvature is Hamiltonian for a Killing vector field. As a geometric PDE, this is quasilinear of fourth order, and no general methods are currently available.

Nevertheless, considerable progress on the existence theory has been made, following the seminal work of Calabi [15] on the non-positive Kähler-Einstein case and the resolution of his famous conjecture by T. Aubin [9] and S-T. Yau [59]. Conjectures going back to Yau [60], G. Tian [56] and S. Donaldson [26] state that the obstruction to the existence of an extremal Kähler metric in the class $\Omega = 2\pi c_1(\mathscr{L})$ of a polarized complex manifold $(M, \mathscr{L})$ should be a purely algebro-geometric "stability condition" on the pair $(M, \mathscr{L})$, and these conjectures may be extended to orbifolds [49]. Defining a precise notion of stability is part of the problem, one candidate being "K-(*poly*)*stability*" [56, 26]: the necessity of K-polystability has been proven for constant scalar curvature metrics [28, 20, 51, 46], and a version of K-polystability relative to a maximal torus of the automorphism group of $(M, \mathscr{L})$, developed by G. Székelyhidi [54, 53], is necessary for the existence of an extremal Kähler metric of non-constant scalar curvature [52].

A major difficulty with the theory is that in practice it is not only difficult to determine whether a given polarized variety admits an extremal Kähler metric—it is also difficult to verify a proposed stability condition. Consequently classes of complex manifolds or orbifolds for which extremality and stability are more tractable play an important role. These examples come in two main flavors: *ruled* and *toric*. Using a construction due to Calabi [16], ruled surfaces and other projective line bundles provide a setting for many explicit extremal Kähler metrics [57, 37, 54, 7]. We refer to these as metrics *of Calabi type*; they admit a Hamiltonian 2-form of order one [4, 5]. The extremality equations reduce to ODEs with explicit polynomial solutions, and stability amounts to a positivity condition on the solution [54, 8]. For toric varieties, in contrast, the extremality equations only reduce to a nonlinear fourth order PDE in the momenta; explicit solutions are hard to find, but the existence theory is well-developed [26, 27] and there is a well-understood notion of "relative K-polystability with respect to toric degenerations" which is widely believed to be equivalent to existence [26, 53, 62, 61]. Explicit examples are largely limited to orthotoric $2m$-orbifolds, which admit a Hamiltonian 2-form of order m and have a convex m-cube (or degeneration) for their rational Delzant polytope [4, 7, 42].

In dimension four, examples and theory come together to provide a fairly complete picture. Extremal Kähler surfaces of Calabi type are locally toric (as the base is a constant curvature Riemann surface) and there are specific results for toric surfaces. The equivalence of existence and (relative) K-polystability is established in [30, 29] in the constant scalar curvature case, while more recent work [18] relaxes this to the assumption that the zero locus of the scalar curvature does not contain a toric divisor.

Our paper is closely related to work of E. Legendre [42], who investigated systematically the extent to which explicit methods resolve the existence problem when the rational Delzant polytope is a convex quadrilateral. Her solution highlighted the role of the *extremal affine function* ζ on the rational Delzant polytope, a combinatorial invariant which pulls back to the scalar curvature in the extremal case. Her main results show that Hamiltonian 2-form methods suffice only for "equipoised" quadrilaterals, for which ζ has equal values at the midpoints of the diagonals. A key ingredient in Legendre's work is the observation that ζ is linear in the inverse lengths of the normals. Using this, she resolved the existence problem for the codimension one family of equipoised quadrilaterals using orthotoric, Calabi type or product metrics.

The theory of Hamiltonian 2-forms in four dimensions [4] implies that these toric metrics are in fact *ambitoric*, i.e., toric with respect to a pair of oppositely oriented but conformally equivalent Kähler metrics. The local classification of ambitoric structures [6] implies that the "regular" examples (i.e., neither a product nor of Calabi type) are determined by a quadratic polynomial q and two functions A, B of one variable. Regular ambitoric structures reduce to orthotoric metrics precisely when q has vanishing discriminant. The extremality conditions for regular ambitoric structures can be explicitly solved with A, B given by quartic polynomials [6], and this generalization suffices to remove the equipoisedness constraint introduced by Legendre.

To prove this, we use, in addition to ambitoric geometry, two further ingredients. The first is an analysis of rational Delzant quadrilaterals building on [41]. We compute the extremal affine function ζ and establish a notion of "temperateness" for polystable quadrilaterals which implies ζ is positive at the midpoints of the diagonals.

The second is the concept of a "factorization structure", which makes precise the separation of variables technique that underpins explicit solutions of geometric PDEs on toric 4-orbifolds. One can hope such an approach will work in $2m$-dimensions when the rational Delzant polytope is a convex m-cube (or degeneration), with the $2m$ facets providing boundary conditions for the m functions of one variable determining the solution. In particular, by the uniqueness of toric extremal Kähler metrics [34], we might *expect* a rational Delzant polytope to select an essentially unique adapted factorization structure for the solution. This is indeed what happens for $m = 2$.

The fruit of this analysis is Theorem 1, which establishes in an explicit way, and for arbitrary quadrilaterals, the conjecture [26] that the existence of an extremal Kähler metric is equivalent to relative K-polystability with respect to toric degenerations. Indeed, we show that temperate quadrilaterals admit a factorization structure which relates the polystability condition directly to the positivity of the quartics A and B appearing in the expression for the extremal ambitoric metrics. This explicitly computable criterion yields new examples both of