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WELL-POSEDNESS FOR THE PRANDTL SYSTEM WITHOUT ANALYTICITY OR MONOTONICITY

BY David GÉRARD-VARET AND Nader MASMOUDI

ABSTRACT. – It has been thought for a while that the Prandtl system is only well-posed under the Oleinik monotonicity assumption or under an analyticity assumption. We show that the Prandtl system is actually locally well-posed for data that belong to the Gevrey class $7/4$ in the horizontal variable x. Our result improves the classical local well-posedness result for data that are analytic in x (that is Gevrey class 1). The proof uses new estimates, based on non-quadratic energy functionals.

RÉSUMÉ. – Il a longtemps été supposé que l'équation de Prandtl n'est bien posée que sous l'hypothèse de monotonie d'Oleinik, ou pour des données analytiques. Nous montrons qu'elle est en fait localement bien posée pour des données appartenant à la classe Gevrey $7/4$ en la variable x. Nous améliorons ainsi le résultat classique d'existence locale de solutions analytiques en la variable x (classe Gevrey 1). La preuve repose sur de nouvelles estimations, faisant appel à des fonctionnelles d'énergie non-quadratiques.

1. Introduction

Our concern in this paper is the well-posedness of the Prandtl system. This system, by now classical, was introduced by Prandtl in 1904 to describe an incompressible flow near a wall, at high Reynolds number. Formally, it is derived from the Navier-Stokes equation with no-slip condition:

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - \varepsilon\Delta\mathbf{u} = 0, & \mathbf{x} \in \Omega, \\ \nabla\cdot\mathbf{u} = 0, & \mathbf{x} \in \Omega, \\ \mathbf{u}|_{\partial\Omega} = 0, & \end{cases} \tag{1.1}$$

that we consider for simplicity in $\Omega := \mathbb{T}\times\mathbb{R}_+$. We recall that $\mathbf{u}(t,\mathbf{x}) = (u,v)(t,x,y)$ is the velocity field of the fluid, and p its pressure field. The parameter $0 < \varepsilon \ll 1$ is the inverse of the Reynolds number. In the limit case $\varepsilon = 0$, one is left formally with the Euler equation,

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for which only the impermeability condition $u \cdot n|_{\partial\Omega} = 0$ can be prescribed. Mathematically, this singular change of boundary condition generates strong gradients of the Navier-Stokes solution $\mathbf{u}^\varepsilon$, as $\varepsilon \to 0$. These gradients correspond to a concentration of the fluid flow in a thin zone near the wall $\partial\Omega$: the so-called *boundary layer*. The understanding of the boundary layer is a great mathematical challenge, that makes the convergence of Navier-Stokes solutions to Euler ones a big open problem, even for smooth data.

To tackle this problem, Prandtl proposed in 1904 an asymptotic model for the flow, based on two different asymptotic expansions of $\mathbf{u}^\varepsilon$, resp. outside and inside the boundary layer:

– outside the boundary layer, no concentration should occur: one should have

$$\mathbf{u}^\varepsilon(t, \mathbf{x}) \sim \mathbf{u}^0(t, \mathbf{x}), \qquad \text{the solution of the Euler equation.}$$

– inside the boundary layer, $\mathbf{u}^\varepsilon$ should exhibit strong gradients, transversally to the boundary: more precisely, the asymptotics suggested by Prandtl is

$$u^\varepsilon(t, x, y) \sim u(t, x, y/\sqrt{\varepsilon}), \quad v^\varepsilon(t, x, y) \sim \sqrt{\varepsilon} v(t, x, y/\sqrt{\varepsilon})$$

where $u = u(t, x, Y)$ and $v = v(t, x, Y)$ are *boundary layer profiles*, depending on a rescaled variable $Y = y/\sqrt{\varepsilon}$, $Y > 0$. Note that the scale $\sqrt{\varepsilon}$ is coherent with the parabolic part of (1.1a).

If we plug the expansion above in (1.1) and keep the leading order terms, we derive the famous Prandtl system (denoting Y instead of y):

$$\text{(1.2)} \qquad \begin{cases} \partial_t u + u\partial_x u + v\partial_y u + \partial_x p - \partial_y^2 u = 0, \\ \partial_y p = 0, \\ \partial_x u + \partial_y v = 0, \\ u|_{y=0} = v|_{y=0} = 0, \quad \lim\limits_{y\to+\infty} u = U, \quad \lim\limits_{y\to+\infty} p = P, \end{cases}$$

where $U(t, x) := u^0(t, x, 0)$ and $P(t, x) := p^0(t, x, 0)$ are the Euler tangential velocity and pressure at the boundary. We refer to [18] for the formal derivation of the Prandtl system. The condition at $y = +\infty$ in (1.2d) is a matching condition near the boundary between the boundary layer flow and the Euler flow (*matched asymptotics*). Note that, combining (1.2b) with the boundary condition on p, we get $p \equiv P$. Hence, the pressure is not an unknown in the Prandtl model: v is obtained in terms of u by integrating the divergence-free condition (1.2c), so that (1.2a) is a scalar evolution equation on u, which is a priori much simpler than the original Navier-Stokes equation.

However, this appealing formal asymptotics raises strong mathematical issues: well-posedness of the limit Prandtl system on one hand, justification of the Prandtl asymptotics of $\mathbf{u}^\varepsilon$ on the other hand. The difficulty comes from numerous underlying fluid instabilities, that can invalidate the Prandtl model: we refer to [11] for a basic presentation of these aspects.

The aim of the present paper is to investigate this stability problem, from a mathematical viewpoint. We shall focus on the limit Prandtl system, namely on its well-posedness. For simplicity, we shall restrict to homogeneous data: $U = P = 0$. Extension of our results to the case of constant U would not raise any problem. Extension to some $U = U(t, x)$

would require some modifications, see [15] for a similar problem. Hence, we consider here the following system:

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_y u - \partial_y^2 u = 0, \\ \partial_x u + \partial_y v = 0, \\ u|_{y=0} = v|_{y=0} = 0, \lim\limits_{y=+\infty} u = 0 \end{cases} \tag{1.3}$$

with initial condition $u|_{t=0} = u_0$.

Before stating our theorem, let us review briefly known results on the existence theory for (1.2). So far, well-posedness has been established in two settings:

– The first results go back to Oleinik [22], who obtained some local well-posedness for *initial data that are monotonic with respect to* y: $U > 0$, $\partial_y u > 0$. For such data, one can use the Crocco transform: in short, using u as an independent variable instead of y and $w := \partial_y u$ as an unknown instead of u, one is left with a nonlinear parabolic equation on w, for which maximum principles are available: see [22] for details. Note that under the extra condition $\partial_x P \leq 0$ (*favorable pressure gradient*), one can go from local to global well-posedness, cf. [26]. From the point of view of physics, this monotonicity assumption is known to be stabilizing: it avoids the *boundary layer separation*, see [11].
– Without monotonicity, well-posedness has been established only locally in time, for *initial data that are analytic with respect to* x. We refer to [25, 17], and to the recent extensions [15, 14]. The assumption of analyticity can be understood as follows. By the divergence-free condition, one obtains $v = -\int_0^y \partial_x u$. Thus, the term $v\,\partial_y u$ in (1.3a) (seen as a functional of u) is first order in x. Moreover, it is not hyperbolic. For instance, let us consider the linearization of the Prandtl equation around a shear flow $\mathbf{u} = (U_s(y), 0)$:

$$\partial_t u + U_s \partial_x u + U_s' v - \partial_y^2 u = 0, \quad \partial_x u + \partial_y v = 0. \tag{1.4}$$

If we freeze the coefficients at some y_0 and compute the dispersion relation, we obtain the growth rate

$$\sigma(k_x, k_y) = U_s'(y_0)\frac{k_x}{k_y} - k_y^2$$

that increases linearly with the wavenumber k_x. This kind of growth rate would prevent any well-posedness result outside the analytic setting.

However, as discussed in [12], this dispersion relation, formally obtained by freezing the coefficients, is misleading: for instance, the inviscid version of Prandtl (that is removing the $\partial_y^2 u$ term) is locally well-posed in C^k, through the method of characteristics.

In the case of the full Prandtl system (1.3), the situation is even more complex, and was addressed recently by the first author and Emmanuel Dormy in article [7] (see also [8]). This article contains a careful study of the linearized system (1.4), *in the case of a non-monotonic base flow* U_s:

$$\exists a, \quad U_s'(a) = 0, \quad U_s''(a) \neq 0.$$