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THE INITIAL VALUE PROBLEM FOR THE BINORMAL FLOW WITH ROUGH DATA

BY VALERIA BANICA AND LUIS VEGA

ABSTRACT. – In this article we consider the initial value problem of the binormal flow with initial data given by curves that are regular except at one point where they have a corner. We prove that under suitable conditions on the initial data a unique regular solution exists for strictly positive and strictly negative times. Moreover, this solution satisfies a weak version of the equation for all times and can be seen as a perturbation of a suitably chosen self-similar solution. Conversely, we also prove that if at time $t = 1$ a small regular perturbation of a self-similar solution is taken as initial condition then there exists a unique solution that at time $t = 0$ is regular except at a point where it has a corner with the same angle as the one of the self-similar solution. This solution can be extended for negative times. The proof uses the full strength of the previous papers [9], [2], [3] and [4] on the study of small perturbations of self-similar solutions. A compactness argument is used to avoid the weighted conditions we needed in [4], as well as a more refined analysis of the asymptotic in time and in space of the tangent and normal vectors.

RÉSUMÉ. – Dans cet article on considère le flot binormal avec données initiales des courbes régulières partout sauf en un point où elles ont un coin. On montre sous des conditions appropriées sur la donnée initiale qu'il existe une unique solution régulière pour des temps strictement positifs et négatifs. De plus, cette solution satisfait le flot binormal en un sens faible et peut être vue comme une perturbation d'une solution auto-similaire bien choisie. Réciproquement, on montre aussi que si à temps $t = 1$ on prend comme donnée initiale une petite perturbation régulière d'une solution auto-similaire, alors il existe une unique solution, qui à temps $t = 0$ est régulière partout sauf en un point où elle a un coin de même angle que celui formé par la solution auto-similaire. Cette solution peut être prolongée aux temps négatifs. La preuve s'appuie sur les résultats des articles précédents [9], [2], [3] et [4] sur l'étude des petites perturbations des solutions auto-similaires. Un argument de compacité est utilisé pour éviter les conditions à poids imposées dans [4], ainsi qu'une analyse plus raffinée des asymptotiques en temps et en espace des vecteurs tangents et normaux.

1. Introduction

We consider the binormal flow equation

$$\chi_t = \chi_x \wedge \chi_{xx}, \tag{1}$$

which is a geometric law for the evolution in time of a curve $\chi(t)$ in $\mathbb{R}^3$, parametrized by arc length x. This model has been proposed in 1906 by Da Rios [7], and rediscovered in 1965 by Arms and Hama [1], as a model for the evolution of a vortex filament in a 3-D inhomogeneous inviscid fluid (see also [21], [20] for the history of this equation). It was also used as a model for vortex filament dynamics in superfluids ([16], [17], [5]). From (1) it follows that the tangent vector $T(t,x)$ satisfies the Schrödinger map equation on the sphere $\mathbb{S}^2$,

$$T_t = T \wedge T_{xx}. \tag{2}$$

Also using the Frenet equations for the tangent T, the normal n, and the binormal b, Equation (1) can be written as

$$\chi_t = cb, \tag{3}$$

with $c(t,x)$ denoting the curvature. Finally, Hasimoto [10] showed that if the curvature $c(t,x)$ does not vanish, then the function

$$\psi(t,x) = c(t,x)e^{i\int_0^x \tau(t,s)\,ds}, \tag{4}$$

that he calls the filament function, solves the focusing cubic non-linear Schrödinger equation (NLS)

$$i\psi_t + \psi_{xx} + \frac{\psi}{2}\left(|\psi|^2 - A(t)\right) = 0 \tag{5}$$

for some real function $A(t)$ that depends on $c(0,t)$ and $\tau(0,t)$. Here τ stands for the torsion. The non-vanishing constraint on the curvature has been removed by Koiso [14], by using another frame instead of Frenet's one.

In view of this link with the nonlinear Schrödinger equation, existence results were given for the initial value problem of the binormal flow with initial data curves with curvature and torsion in high order Sobolev spaces ([10], [14], [8]). The case of less regular closed curves was considered recently by Jerrard and Smets by using a weak version of the binormal flow ([12], [13]). Let us mention also that stability of various types of particular solutions of the binormal flow is a subject of current research (see for instance [6], [15] and the references therein). Also, to emphasize the great complexity of the binormal flow, we recall that in the case of closed curves, various aspects of evolutions of knotted vortices by the binormal flow are studied using geometric and topological methods (as an example, see [18] and the references therein).

We are interested in solutions of (1) that at a given time are regular except at a point where they have a corner. One can use the invariance of the equation under translations in time and in space and assume without loss of generality that the time is $t=0$ and the corner is located at the origin $(0,0,0)$. Let us also note that the equation is reversible in time. This is because if $\chi(t,x)$ is a solution so is $\chi(-t,-x)$.

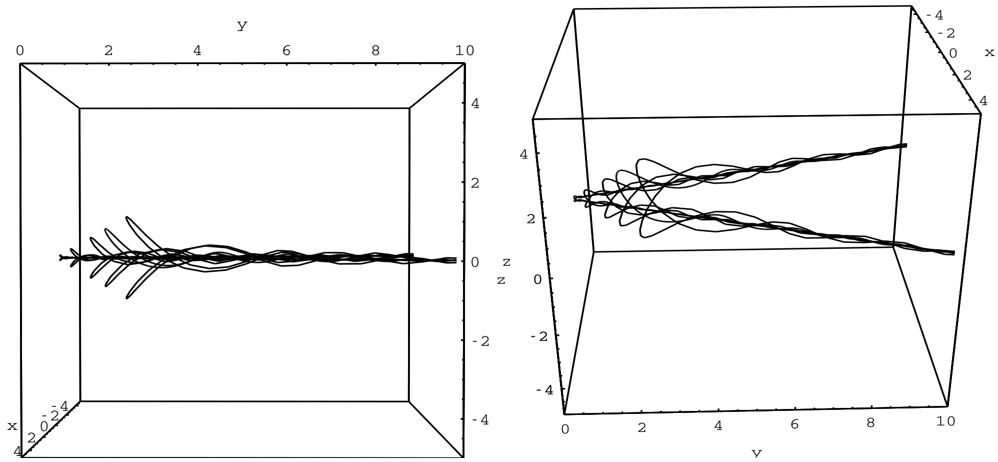

FIGURE 1. Self-similar solution for negative and positive times, from two different angles.

One relevant class of solutions are the self-similar ones, i.e., those that can be written as

$$\chi(t,x)=\sqrt{t}\,G\left(\frac{x}{\sqrt{t}}\right)$$

for some appropriate G. These solutions have been investigated first by physicists in the 80's. In fact it is rather easy to see that, modulo rotations, self-similar solutions are a family of curves χ_a parametrized by $a\in\mathbb{R}^{+*}$, such that curvature and torsion of $\chi_a(t)$ at x are $\frac{a}{\sqrt{t}}$ and $\frac{x}{2t}$ respectively ([16], [17], [5]). From this it is not complicated to conclude that $\chi_a(0)$ has a corner at $(0,0,0)$. This fact, together with a characterization and detailed asymptotic of the self-similar solutions was proved in [9]. We reformulate part of Theorem 1 of [9] as follows. Details will be given in the next section.

THEOREM 1.1 (Description of self-similar solutions [9]). – *Let A^+ and A^- be any two distinct non-colinear unitary vectors in $\mathbb{R}^3$. Then, there exists a unique self-similar solution χ for positive times with initial data at time $t=0$*

$$\chi(0,x)=\begin{cases}A^+x,\ x\geq 0,\\ A^-x,\ x\leq 0.\end{cases}$$

All self-similar solutions are described in this way. Moreover, if we denote $a\in\mathbb{R}^{+*}$ such that $\sin\frac{(\widehat{A^+,-A^-})}{2}=e^{-\pi\frac{a^2}{2}}$, $\frac{a}{\sqrt{t}}$ and $\frac{x}{2t}$ are respectively the curvature and the torsion of the curve $\chi(t,x)$. Also, there exist two complex vectors $B^\pm$ orthonormal to $A^\pm$ such that

$$B^\pm=\lim_{x\to\pm\infty}(n+ib)(t,x)e^{i\int_0^x\tau(t,s)ds}e^{-ia^2\log\sqrt{t}+ia^2\log|x|}.$$

Up to a rotation, the coordinates of $A^\pm$ and $B^\pm$ are given explicitly in terms of Gamma functions involving the parameter a (see formulas (55), (57), (47), (48), and (69) in [9]).

Finally, we want to remark that the solution given in the above theorem can be continued as a self-similar solution in a unique way for negative times. This is done as follows. If χ in the Theorem 1.1 exists for negative times, then $\chi^*(t,x)=\chi(-t,-x)$ with $t>0$ is a solution for