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Cohomology jump loci of quasi-projective varieties

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COHOMOLOGY JUMP LOCI
OF QUASI-PROJECTIVE VARIETIES

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Abstract. – We prove that the cohomology jump loci in the space of rank one local systems over a smooth quasi-projective variety are finite unions of torsion translates of subtori. The main ingredients are a recent result of Dimca-Papadima, some techniques introduced by Simpson, together with properties of the moduli space of logarithmic connections constructed by Nitsure and Simpson.

Résumé. – Dans cet article, on montre que les lieux de saut dans l’espace de systèmes locaux de rang un sur une variété lisse quasi-projective sont des réunions finies de subtores translatées par des éléments de torsion. Pour cela, nous utilisons un résultat récent de Dimca-Papadima, certaines techniques introduites par Simpson, ainsi que des propriétés de l’espace de moduli pour les connexions logarithmiques construit par Nitsure et Simpson.

1. Introduction

Let $X$ be a connected, finite-type CW-complex. Define

$$M_B(X) = \text{Hom}(\pi_1(X), \mathbb{C}^*)$$

to be the variety of $\mathbb{C}^*$ representations of $\pi_1(X)$. Then $M_B(X)$ is a direct product of $(\mathbb{C}^*)^{b_1(X)}$ and a finite Abelian group. For each point $\rho \in M_B(X)$, there exists a unique rank one local system $L_\rho$, whose monodromy representation is $\rho$. The cohomology jump loci of $X$ are the natural strata

$$\Sigma^i_k(X) = \{\rho \in M_B(X) \mid \dim_{\mathbb{C}} H^i(X, L_\rho) \geq k\}.$$ 

$\Sigma^i_k(X)$ is a Zariski closed subset of $M_B(X)$. A celebrated result of Simpson says that if $X$ is a smooth projective variety defined over $\mathbb{C}$, then $\Sigma^i_k(X)$ is a union of torsion translates of subtori of $M_B(X)$.

In this paper, we generalize Simpson’s result to quasi-projective varieties.

Theorem 1.1. – Suppose $U$ is a smooth quasi-projective variety defined over $\mathbb{C}$. Then $\Sigma^i_k(U)$ is a finite union of torsion translates of subtori of $M_B(U)$.
When \( U \) is compact, the theorem is proved in [7, 8], [1], [15], with the strongest form appearing in the latter. When \( b_1(U) = 0 \), Arapura [2] showed that \( \Sigma_k^1(U) \) are union of translates of subtori. The case of unitary rank one local systems on \( U \) has been considered in [3]. Dimca and Papadima were able to prove the following:

**Theorem 1.2** ([6, Theorem C]). - Under the same assumption as Theorem 1.1, every irreducible component of \( \Sigma_k^1(U) \) containing \( 1 \in M_B(U) \) is a subtorus.

The proof of this result reduces to the study of the infinitesimal deformations with cohomology constraints of the trivial local system. These are governed in general by infinite-dimensional models. In [6] it is shown that, in this case, the finite-dimensional Gysin model due to Morgan provides the necessary linear algebra description for the infinitesimal deformations.

The result of Dimca and Papadima serves as a key ingredient of our theorem. In Section 2, we will show that each irreducible component of \( \Sigma_k^1(U) \) contains a torsion point. Then, in Section 3, we will see that, thanks to Theorem 1.2, having a torsion point on an irreducible component of \( \Sigma_k^1(U) \) forces this component to be a translate of subtorus.

There are two other proofs of Simpson’s theorem: one via positive characteristic methods [11], and one via D-modules [13, 12]. However, in this paper we follow the original approach of Simpson. There are no analogous results for higher rank local systems even in the projective case.

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## 2. Torsion points on the cohomology jump loci

Let \( X \) be a smooth complex projective variety, and let \( D = \sum_{\lambda=1}^n \lambda D_\lambda \) be a simple normal crossing divisor on \( X \) with irreducible components \( D_\lambda \). Let \( U = X - D \). Thanks to Hironaka’s theorem on resolution of singularities, every smooth quasi-projective variety \( U \) can be realized in this way. The goal of this section is to prove the following:

**Theorem 2.1.** - Each irreducible component of \( \Sigma_k^1(U) \) contains a torsion point.

First, we want to reduce to the case when \( X \) and each \( D_\lambda \) are defined over \( \bar{\mathbb{Q}} \). This can be done using a technique which we have learnt from the proof of [15, Theorem 4.1]. We reproduce it here.

We can assume \( X \) and each \( D_\lambda \) to be defined over a subring \( O \) of \( \mathbb{C} \), which is finitely generated over \( \mathbb{Q} \). Denote the embedding of \( O \) to \( \mathbb{C} \) by \( \sigma : O \to \mathbb{C} \). Each ring homomorphism \( O \to \mathbb{C} \) corresponds to a point in \( \text{Spec}(O)(\mathbb{C}) \). Denote by \( X^0 \) and \( D_\lambda^0 \) the schemes over \( \text{Spec}(O) \) which give rise to \( X \) and \( D_\lambda \) respectively after tensoring with \( \mathbb{C} \), that is \( X = X^0 \times_{\text{Spec}(O)} \text{Spec}(\mathbb{C}) \) and \( D_\lambda = D_\lambda^0 \times_{\text{Spec}(O)} \text{Spec}(\mathbb{C}) \). By possibly replacing \( O \) by \( O[1/h] \) for some \( h \in O \), we can assume \( X^0 \) and every \( D_\lambda^0 \) are smooth over \( \text{Spec}(O) \), and all the intersections of \( D_\lambda^0 \)'s are transverse. Since each connected component of \( \text{Spec}(O)(\mathbb{C}) \) contains a \( \bar{\mathbb{Q}} \) point, there exists a point \( P \in \text{Spec}(O)(\bar{\mathbb{Q}}) \), and a continuous path from \( \sigma \in \text{Spec}(O)(\mathbb{C}) \) to \( P \) in \( \text{Spec}(O)(\mathbb{C})^{\text{top}} \). Then, according to Thom’s First Isotopy Lemma [5, Ch. 1, Theorem 3.5], \( X^0(\mathbb{C}) \) together with its strata given by the \( D_\lambda^0(\mathbb{C}) \), is a topologically
locally trivial fibration in the stratified sense over $\text{Spec}(O)(\mathbb{C})^{\text{top}}$. In particular, letting $X'$ and $D'_k$ be the corresponding fibers over $P$, transporting along the path gives an isomorphism $(X - D)^{\text{top}} \cong (X' - D'_k)^{\text{top}}$. Recall that $M_B(U)$ and $\Sigma_k(U)$ depend only on the topology of $U$. Hence replacing $U = X - D$ by $U' = X' - D'$, we may assume that $X$ and each $D_k$ are defined over $\overline{\mathbb{Q}}$.

Next, we introduce the other side of the story, namely the logarithmic flat bundles on $(X, D)$. A logarithmic flat bundle on $(X, D)$ consists of a vector bundle $E$ on $X$, and a logarithmic connection $\nabla : E \to E \otimes \Omega_X^1(\log D)$, satisfying the integrability condition $\nabla^2 = 0$. Given a logarithmic flat bundle $(E, \nabla)$, the flat sections of $E$ on $U$ (by which we will always mean on $U^{\text{top}}$) form a local system. And conversely, given any local system $L$ on $U$ (by which, as in the introduction, we will always mean a local system on $U^{\text{top}}$), it is always obtained from some logarithmic flat bundle $(E, \nabla)$. However, different logarithmic flat bundles may give the same local system. This correspondence between local systems on $U$ and logarithmic flat bundles on $(X, D)$ is very well understood (e.g., [4], [14], [9]).

For a vector bundle $E$ on $X$, the structure of a logarithmic flat bundle $(E, \nabla)$ on $(X, D)$ is the same as a $\mathcal{D}_X(\log D)$-module structure on $E$, where $\mathcal{D}_X(\log D)$ is the sheaf of logarithmic differentials.

Nitsure [10] and Simpson [16] constructed coarse moduli spaces, for Jordan-equivalence classes of semistable $\Lambda$-modules which are $\mathcal{D}_X$-coherent and torsion free, where $\Lambda$ is a sheaf of rings of differential operators. The two examples of $\Lambda$ which we are concerned with are $\mathcal{D}_X$, the usual sheaf of differential operators on $X$, and $\mathcal{D}_X(\log D)$, the sheaf of logarithmic differentials. We denote by $M_{\text{DR}}(X)$ and $M_{\text{DR}}(X/D)$ the moduli space of rank one $\mathcal{D}_X$-modules and the moduli space of rank one $\mathcal{D}_X(\log D)$-modules, respectively. In the rank one case, semistable is the same as stable and this condition is automatic as is the locally free condition, and Jordan-equivalence is the same as isomorphic. Thus, the points of $M_{\text{DR}}(X)$ and $M_{\text{DR}}(X/D)$ correspond to isomorphism classes of flat, respectively, logarithmic flat line bundles. Since we did not put any condition on the Chern class of the underlying line bundles, in general $M_{\text{DR}}(X/D)$ has infinitely many connected components. $M_{\text{DR}}(X)$, $M_{\text{DR}}(X/D)$, $M_B(X)$ and $M_B(U)$ are all algebraic groups, except $M_{\text{DR}}(X/D)$ may not be of finite type.

The diagram of Fig. 1 (p. 230) plays an essential role in our proof.

Let us first explain how the arrows are defined. Since every $\mathcal{D}_X$-module is naturally a $\mathcal{D}_X(\log D)$-module, there is a natural embedding $M_{\text{DR}}(X) \hookrightarrow M_{\text{DR}}(X/D)$. On the other hand, the embedding $U \hookrightarrow X$ induces a surjective map on the fundamental group $\pi_1(U) \twoheadrightarrow \pi_1(X)$. Composing this map with the representations, we have $M_B(X) \hookrightarrow M_B(U)$. For every rank one logarithmic flat bundle $(E, \nabla)$, taking the residue along each $D_k$ is the map $\text{res}$. In other words, $\text{res}((E, \nabla)) = \{\text{res}_{D_k}(\nabla)\}_{1 \leq k \leq n}$. Around each $D_k$, we can take a small loop $\gamma_k$. The map $\text{ev}$ is the evaluation at the loops $\gamma_k$. More precisely $\text{ev}(\rho) = \{\rho(\gamma_k)\}_{1 \leq k \leq n}$.

For the horizontal arrows, $R^H : M_{\text{DR}}(X) \to M_B(X)$ is taking the monodromy representations for flat bundles. Since every logarithmic flat bundle on $(X, D)$ restricts to a flat bundle on $U$, taking the monodromy representation on $U$ is $R^H : M_{\text{DR}}(X/D) \to M_B(U)$. The map $\text{exp} : \mathbb{C}^n \to (\mathbb{C}^*)^n$ is component-wise defined to be multiplying by $2\pi \sqrt{-1}$, then taking exponential. On $M_{\text{DR}}(X/D)$, there are some special elements. Let $(\mathcal{O}_X, d)$ be the