ISSN 0012-9593

quatrième série - tome 48 *fascicule 1* *janvier-février 2015*

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

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SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

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Édition / *Publication*

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Tarifs

Europe : 515 €. Hors Europe : 545 €. Vente au numéro : 77 €.

ISSN 0012-9593

Directeur de la publication : Marc Peigné
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4^{e} série, t. 48, 2015, p. 227 à 236

COHOMOLOGY JUMP LOCI OF QUASI-PROJECTIVE VARIETIES

BY NERO BUDUR AND BOTONG WANG

ABSTRACT. – We prove that the cohomology jump loci in the space of rank one local systems over a smooth quasi-projective variety are finite unions of torsion translates of subtori. The main ingredients are a recent result of Dimca-Papadima, some techniques introduced by Simpson, together with properties of the moduli space of logarithmic connections constructed by Nitsure and Simpson.

RÉSUMÉ. – Dans cet article, on montre que les lieux de saut dans l'espace de systèmes locaux de rang un sur une variété lisse quasi-projective sont des réunions finies de subtores translatées par des éléments de torsion. Pour cela, nous utilisons un résultat récent de Dimca-Papadima, certaines techniques introduites par Simpson, ainsi que des propriétés de l'espace de moduli pour les connexions logarithmiques construit par Nitsure et Simpson.

1. Introduction

Let X be a connected, finite-type CW-complex. Define

$$\mathbf{M}_{\mathbf{B}}(X) = \mathrm{Hom}(\pi_1(X), \mathbb{C}^*)$$

to be the variety of $\mathbb{C}^*$ representations of $\pi_1(X)$. Then $\mathbf{M}_{\mathbf{B}}(X)$ is a direct product of $(\mathbb{C}^*)^{b_1(X)}$ and a finite Abelian group. For each point $\rho \in \mathbf{M}_{\mathbf{B}}(X)$, there exists a unique rank one local system L_ρ, whose monodromy representation is ρ. The *cohomology jump loci* of X are the natural strata

$$\Sigma^i_k(X) = \{\rho \in \mathbf{M}_{\mathbf{B}}(X) \mid \dim_{\mathbb{C}} H^i(X, L_\rho) \geq k\}.$$

$\Sigma^i_k(X)$ is a Zariski closed subset of $\mathbf{M}_{\mathbf{B}}(X)$. A celebrated result of Simpson says that if X is a smooth projective variety defined over $\mathbb{C}$, then $\Sigma^i_k(X)$ is a union of torsion translates of subtori of $\mathbf{M}_{\mathbf{B}}(X)$.

In this paper, we generalize Simpson's result to quasi-projective varieties.

THEOREM 1.1. – *Suppose U is a smooth quasi-projective variety defined over $\mathbb{C}$. Then $\Sigma^i_k(U)$ is a finite union of torsion translates of subtori of $\mathbf{M}_{\mathbf{B}}(U)$.*

When U is compact, the theorem is proved in [7, 8], [1], [15], with the strongest form appearing in the latter. When $b_1(\bar{U}) = 0$, Arapura [2] showed that $\Sigma^i_k(U)$ are union of translates of subtori. The case of unitary rank one local systems on U has been considered in [3]. Dimca and Papadima were able to prove the following:

THEOREM 1.2 ([6, Theorem C]). – *Under the same assumption as Theorem 1.1, every irreducible component of $\Sigma^i_k(U)$ containing $\mathbf{1} \in \mathbf{M_B}(U)$ is a subtorus.*

The proof of this result reduces to the study of the infinitesimal deformations with cohomology constraints of the trivial local system. These are governed in general by infinite-dimensional models. In [6] it is shown that, in this case, the finite-dimensional Gysin model due to Morgan provides the necessary linear algebra description for the infinitesimal deformations.

The result of Dimca and Papadima serves as a key ingredient of our theorem. In Section 2, we will show that each irreducible component of $\Sigma^i_k(U)$ contains a torsion point. Then, in Section 3, we will see that, thanks to Theorem 1.2, having a torsion point on an irreducible component of $\Sigma^i_k(U)$ forces this component to be a translate of subtorus.

There are two other proofs of Simpson's theorem: one via positive characteristic methods [11], and one via D-modules [13, 12]. However, in this paper we follow the original approach of Simpson. There are no analogous results for higher rank local systems even in the projective case.

Acknowledgement. The first author was partially supported by the NSA, the Simons Foundation grant 245850, and the BOF-OT KU Leuven grant.

2. Torsion points on the cohomology jump loci

Let X be a smooth complex projective variety, and let $D = \sum_{\lambda=1}^n D_\lambda$ be a simple normal crossing divisor on X with irreducible components D_λ. Let $U = X - D$. Thanks to Hironaka's theorem on resolution of singularities, every smooth quasi-projective variety U can be realized in this way. The goal of this section is to prove the following:

THEOREM 2.1. – *Each irreducible component of $\Sigma^i_k(U)$ contains a torsion point.*

First, we want to reduce to the case when X and each D_λ are defined over $\bar{\mathbb{Q}}$. This can be done using a technique which we have learnt from the proof of [15, Theorem 4.1]. We reproduce it here.

We can assume X and each D_λ to be defined over a subring O of $\mathbb{C}$, which is finitely generated over $\mathbb{Q}$. Denote the embedding of O to $\mathbb{C}$ by $\sigma : O \to \mathbb{C}$. Each ring homomorphism $O \to \mathbb{C}$ corresponds to a point in $\mathrm{Spec}(O)(\mathbb{C})$. Denote by X^0 and D^0_λ the schemes over $\mathrm{Spec}(O)$ which give rise to X and D_λ respectively after tensoring with $\mathbb{C}$, that is $X = X^0 \times_{\mathrm{Spec}(O)} \mathrm{Spec}(\mathbb{C})$ and $D_\lambda = D^0_\lambda \times_{\mathrm{Spec}(O)} \mathrm{Spec}(\mathbb{C})$. By possibly replacing O by $O[\frac{1}{h}]$ for some $h \in O$, we can assume X^0 and every D^0_λ are smooth over $\mathrm{Spec}(O)$, and all the intersections of D^0_λ's are transverse. Since each connected component of $\mathrm{Spec}(O)(\mathbb{C})$ contains a $\bar{\mathbb{Q}}$ point, there exists a point $P \in \mathrm{Spec}(O)(\bar{\mathbb{Q}})$, and a continuous path from $\sigma \in \mathrm{Spec}(O)(\mathbb{C})$ to P in $\mathrm{Spec}(O)(\mathbb{C})^{\mathrm{top}}$. Then, according to Thom's First Isotopy Lemma [5, Ch. 1, Theorem 3.5], $X^0(\mathbb{C})$ together with its strata given by the $D^0_\lambda(\mathbb{C})$, is a topologically locally trivial fibration in the stratified sense over $\mathrm{Spec}(O)(\mathbb{C})^{\mathrm{top}}$. In particular, letting X' and D'_λ be the corresponding fibers over P, transporting along the path gives an isomorphism $(X - D)^{\mathrm{top}} \cong (X' - D')^{\mathrm{top}}$. Recall that $\mathbf{M}_{\mathrm{B}}(U)$ and $\Sigma^i_k(U)$ depend only on the topology of U. Hence replacing $U = X - D$ by $U' = X' - D'$, we may assume that X and each D_λ are defined over $\bar{\mathbb{Q}}$.

Next, we introduce the other side of the story, namely the logarithmic flat bundles on (X, D). A logarithmic flat bundle on (X, D) consists of a vector bundle E on X, and a logarithmic connection $\nabla : E \to E \otimes \Omega^1_X(\log D)$, satisfying the integrability condition $\nabla^2 = 0$. Given a logarithmic flat bundle (E, ∇), the flat sections of E on U (by which we will always mean on U^{top}) form a local system. And conversely, given any local system L on U (by which, as in the introduction, we will always mean a local system on U^{top}), it is always obtained from some logarithmic flat bundle (E, ∇). However, different logarithmic flat bundles may give the same local system. This correspondence between local systems on U and logarithmic flat bundles on (X, D) is very well understood (e.g., [4], [14], [9]).

For a vector bundle E on X, the structure of a logarithmic flat bundle (E, ∇) on (X, D) is the same as a $\mathcal{D}_X(\log D)$-module structure on E, where $\mathcal{D}_X(\log D)$ is the sheaf of logarithmic differentials.

Nitsure [10] and Simpson [16] constructed coarse moduli spaces, which are separated quasi-projective schemes, for Jordan-equivalence classes of semistable Λ-modules which are $\mathcal{O}_X$-coherent and torsion free, where Λ is a sheaf of rings of differential operators. The two examples of Λ which we are concerned with are $\mathcal{D}_X$, the usual sheaf of differential operators on X, and $\mathcal{D}_X(\log D)$, the sheaf of logarithmic differentials. We denote by $\mathbf{M}_{\mathrm{DR}}(X)$ and $\mathbf{M}_{\mathrm{DR}}(X/D)$ the moduli space of rank one $\mathcal{D}_X$-modules and the moduli space of rank one $\mathcal{D}_X(\log D)$-modules, respectively. In the rank one case, semistable is the same as stable and this condition is automatic as is the locally free condition, and Jordan-equivalence is the same as isomorphic. Thus, the points of $\mathbf{M}_{\mathrm{DR}}(X)$ and $\mathbf{M}_{\mathrm{DR}}(X/D)$ correspond to isomorphism classes of flat, respectively, logarithmic flat line bundles. Since we did not put any condition on the Chern class of the underlying line bundles, in general $\mathbf{M}_{\mathrm{DR}}(X/D)$ has infinitely many connected components. $\mathbf{M}_{\mathrm{DR}}(X)$, $\mathbf{M}_{\mathrm{DR}}(X/D)$, $\mathbf{M}_{\mathrm{B}}(X)$ and $\mathbf{M}_{\mathrm{B}}(U)$ are all algebraic groups, except $\mathbf{M}_{\mathrm{DR}}(X/D)$ may not be of finite type.

The diagram of Fig. 1 (p. 230) plays an essential role in our proof.

Let us first explain how the arrows are defined. Since every $\mathcal{D}_X$-module is naturally a $\mathcal{D}_X(\log D)$-module, there is a natural embedding $\mathbf{M}_{\mathrm{DR}}(X) \hookrightarrow \mathbf{M}_{\mathrm{DR}}(X/D)$. On the other hand, the embedding $U \hookrightarrow X$ induces a surjective map on the fundamental group $\pi_1(U) \to \pi_1(X)$. Composing this map with the representations, we have $\mathbf{M}_{\mathrm{B}}(X) \hookrightarrow \mathbf{M}_{\mathrm{B}}(U)$. For every rank one logarithmic flat bundle (E, ∇), taking the residue along each D_λ is the map res. In other words, $\mathrm{res}((E, \nabla)) = \{\mathrm{res}_{D_\lambda}(\nabla)\}_{1 \le \lambda \le n}$. Around each D_λ, we can take a small loop γ_λ. The map ev is the evaluation at the loops γ_λ. More precisely $\mathrm{ev}(\rho) = \{\rho(\gamma_\lambda)\}_{1 \le \lambda \le n}$.

For the horizontal arrows, $RH : \mathbf{M}_{\mathrm{DR}}(X) \to \mathbf{M}_{\mathrm{B}}(X)$ is taking the monodromy representations for flat bundles. Since every logarithmic flat bundle on (X, D) restricts to a flat bundle on U, taking the monodromy representation on U is $RH : \mathbf{M}_{\mathrm{DR}}(X/D) \to \mathbf{M}_{\mathrm{B}}(U)$. The map $\exp : \mathbb{C}^n \to (\mathbb{C}^*)^n$ is component-wise defined to be multiplying by $2\pi\sqrt{-1}$, then taking exponential. On $\mathbf{M}_{\mathrm{DR}}(X/D)$, there are some special elements. Let $(\mathcal{O}_X, d)$ be the