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Mihai DAMIAN

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Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
annales@ens.fr

Édition / *Publication*

Société Mathématique de France
Institut Henri Poincaré
11, rue Pierre et Marie Curie
75231 Paris Cedex 05
Tél. : (33) 01 44 27 67 99
Fax : (33) 01 40 46 90 96

Abonnements / *Subscriptions*

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 09
Fax : (33) 04 91 41 17 51
email : smf@smf.univ-mrs.fr

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ON THE TOPOLOGY OF MONOTONE LAGRANGIAN SUBMANIFOLDS

BY Mihai DAMIAN

ABSTRACT. – We find new obstructions on the topology of closed monotone Lagrangian submanifolds of $\mathbf{C}^n$ under some hypotheses on the homology of their universal cover. In particular we show that nontrivial connected sums of manifolds of odd dimensions do not admit monotone Lagrangian embeddings into $\mathbf{C}^n$ whereas some of these examples are known to admit usual Lagrangian embeddings: the question of the existence of a monotone embedding for a given Lagrangian in $\mathbf{C}^n$ was open. In dimension three we get as a corollary that the only orientable Lagrangians in $\mathbf{C}^3$ are products $\mathbf{S}^1 \times \Sigma$. The main ingredient of our proofs is the lifted Floer homology theory which we developed in [13].

RÉSUMÉ. – Nous trouvons de nouvelles obstructions sur la topologie des sous-variétés lagrangiennes compactes monotones de $\mathbf{C}^n$ sous certaines hypothèses sur l'homologie de leur revêtement universel. Nous montrons en particulier que les sommes connexes non-triviales de variétés compactes de dimension impaire n'admettent pas de plongement lagrangien monotone dans $\mathbf{C}^n$: la question de l'existence de tels plongements était ouverte. En dimension trois nous obtenons comme corollaire que les seules sous-variétés lagrangiennes compactes monototones et orientables de $\mathbf{C}^3$ sont les produits $\mathbf{S}^1 \times \Sigma$. L'outil principal de nos preuves est l'homologie de Floer relevée que nous avons définie en [13].

1. Introduction

1.1. Motivation

The starting point of this paper is the following general question:

What can be said about the topology of a closed Lagrangian $L \subset \mathbf{C}^n$?

A celebrated theorem of M. Gromov [20] asserts that such a submanifold cannot be simply connected and it is easy to prove that it also has to satisfy $\chi(L) = 0$ if it is orientable (this equality only holds modulo 4 in general [2]), but can one say more? The answer is affirmative in low dimensions; for instance when $n = 2$ we know precisely the surfaces admitting Lagrangian embeddings: the only orientable one is the torus, and the non-orientable Lagrangians are those whose Euler characteristic is a multiple of 4 [19], with the exception of the Klein bottle [27, 22]. We also know many examples of closed Lagrangians in $\mathbf{C}^3$: a recent theorem of T. Ekholm, Y. Eliashberg, E. Murphy and I. Smith [15] (2013) asserts:

THEOREM 1.1. – *If L is closed and orientable of dimension three then there is a Lagrangian embedding $L\#(\mathbf{S}^1 \times \mathbf{S}^2) \to \mathbf{C}^3$.*

On the other hand, if L is prime (i.e., not decomposable into a nontrivial connected sum) there is a statement of K. Fukaya [17] (2006) claiming a strong topological rigidity:

CLAIM. – *If L is a three-manifold which is closed orientable and prime then it admits a Lagrangian embedding into $\mathbf{C}^3$ if and only if it is diffeomorphic to a product $\mathbf{S}^1 \times \Sigma$.*

In greater dimensions it is of course too optimistic to expect such strong results of the topology of Lagrangian submanifolds. There are not many examples of Lagrangians available and we do not know if this lack of examples comes from some strong topological restrictions still to be discovered. One of the first papers concerned with the topology of closed Lagrangians in $\mathbf{C}^n$ and the possible examples was written by M. Audin [2] in 1988. The author observed that all the known orientable Lagrangian submanifolds have a common feature: they fiber over the circle. This led to the natural question of the existence of other examples. Few years later, in 1991, L. Polterovich [26] gave a negative answer to this question: he constructed a lot of new examples of Lagrangian submanifolds starting from Lagrangian immersions and replacing neighborhoods of the double points by 1-handles. The resulting Lagrangians are often connected sums, which do not fiber over the circle. Indeed, it can be proved more generally that a manifold whose fundamental group is a nontrivial free product, cannot fiber over the circle (see [12], Prop.2.3 for instance). Here is the theorem of Polterovich [26]:

THEOREM 1.2. – *Let $P = \mathbf{S}^{n-1} \times \mathbf{S}^1$ and Q the manifold obtained from $\mathbf{S}^{n-1} \times [0, 1]$ after gluing the points $(x, 1)$ and $(\tau x, -1)$ where τ is the standard reversing orientation involution on $\mathbf{S}^{n-1}$. Then:*

a) *Let L_1 and L_2 be closed connected manifolds admitting Lagrangian embeddings into $\mathbf{C}^n$. Then $L_1\#L_2\#Q$ admits a Lagrangian embedding into $\mathbf{C}^n$. Moreover $L_1\#L_2\#P$ admits a Lagrangian embedding into $\mathbf{C}^n$ if n is odd.*
b) *Let L be a closed connected manifold admitting a Lagrangian immersion into $\mathbf{C}^n$. Then $L\#kQ$ admits a Lagrangian embedding into $\mathbf{C}^n$ for some non-negative integer k. Moreover $L\#kP$ admits a Lagrangian embedding into $\mathbf{C}^n$ if n is odd.*

L. Polterovich also points out that any closed orientable 3-manifold admits a Lagrangian immersion in $\mathbf{C}^3$. This gives many orientable Lagrangians (of odd dimension) not fibering over the circle, as $\mathbf{T}^{2n+1}\#\mathbf{T}^{2n+1}\#P$, $P\#P\#P$, $\mathbf{T}^{2n+1}\#P\#P$ etc. Taking cartesian products, we get examples of even dimension too.

The development of Floer homology techniques led to an intensive study of a special class of Lagrangian submanifolds called monotone. Recall that a Lagrangian submanifold $L \subset (M, \omega)$ is defined to be monotone if the morphism $I_\omega : \pi_2(M, L) \to \mathbf{R}$ given by the symplectic area and the morphism $I_\mu : \pi_2(M, L) \to \mathbf{Z}$ defined by the Maslov index [1] are positively proportional. Monotone submanifolds are known to be more rigid with respect to Lagrangian intersections. Also some constraints on their topology were established

([5, 6], [7, 10], [8]) but these topological properties are not known to be specific to monotone Lagrangians. It is therefore very tempting to reformulate Audin's question in these terms:

QUESTION 1. – Let $L \subset \mathbf{C}^n$ be a closed orientable monotone Lagrangian. Does L fiber over $\mathbf{S}^1$?

In view of Polterovich's examples another natural question related to the previous is the following:

QUESTION 2. – Let $L \subset \mathbf{C}^n$ be a closed (orientable) Lagrangian. Does L also admit a monotone Lagrangian embedding into $\mathbf{C}^n$?

The aim of this paper is to show that there is indeed a topological rigidity specific to monotone Lagrangian submanifolds. The answer to Question 2 is negative, and many of Polterovich's examples cannot be embedded as monotone Lagrangians. We are able to prove a weaker version of the assertion of Question 1. ("stable fibration" instead of fibration), but only under a hypothesis related to the holomorphic disks of Maslov index equal to 2, with boundary in L (in particular the minimal Maslov number of these submanifolds is supposed to be $N_L = 2$). Note that a topological constraint for monotone Lagrangians with $N_L = n$ was established in our previous paper ([13], Th.1.7). Recall that the minimal Maslov number $N_L \in \mathbf{N}$ is defined to be the positive generator of $\mathrm{Im}(I_\mu)$.

The case of 3-Lagrangians is easier to deal with. In this case we prove that the answer to Question 1 is positive; moreover we have $L = \mathbf{S}^1 \times \Sigma$, as in Fukaya's statement. As far as we know this question is completely open in greater dimensions.

1.2. Main results

All our results are valid for monotone symplectic manifolds M which are closed or convex at infinity and for closed orientable Lagrangians $L \subset M$ with the property that $\phi(L) \cap L = \varnothing$, for some Hamiltonian diffeomorphism ϕ: these submanifolds are called displaceable. Of course any closed submanifold in $\mathbf{C}^n$ or more generally in $M = \mathbf{C} \times W$, is displaceable. We suppose without restricting generality that all the Lagrangian submanifolds we consider are of dimension greater than three. Here is our first result:

THEOREM 1.3. – *Let $L \subset M$ be a closed orientable Lagrangian which is displaceable. Denote by $\widetilde{L}$ its universal cover. Suppose that*

(a) $H_{2i+1}(\widetilde{L}, \mathbf{Z}/2) = 0$ *for all* i.
(b) $\pi_1(L) = G_1 * G_2$ *with* $\mathrm{rk}(G_j/[G_j, G_j]) \neq 0$ *for* $j = 1, 2$.
Then L is not monotone.

Here $*$ denotes the free product of two groups and "rk" is the rank, i.e., the minimal number of generators of the free part of a finitely generated abelian group. As one can easily check, in odd dimension the hypothesis (a) is preserved by connected sums. We have:

THEOREM 1.4. – *Let $L^n \subset M^{2n}$ be a closed orientable Lagrangian of odd dimension which is displaceable. Suppose that $L = L_1 \# L_2$ and the manifolds L_j satisfy the hypothesis* (a) *of 1.3. Then L is not monotone.*