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DIFFRACTION FROM CONORMAL SINGULARITIES

BY MAARTEN DE HOOP, GUNTHER UHLMANN AND ANDRÁS VASY

ABSTRACT. – In this paper we show that for metrics with conormal singularities that correspond to class $C^{1,\alpha}$, $\alpha > 0$, the reflected wave is more regular than the incident wave in a Sobolev sense. This is helpful in the analysis of the multiple scattering series since higher order terms can be effectively 'peeled off'.

RÉSUMÉ. – Dans cet article nous montrons que, pour une métrique avec des singularités conormales qui correspondent à la classe $C^{1,\alpha}$, $\alpha > 0$, l'onde réfléchie est plus régulière que l'onde incidente dans un sens Sobolev. Cela s'avère utile à l'analyse des séries de diffusion multiple, les termes d'ordres les plus élevés pouvant être ôtés de manière effective.

1. Introduction

In this paper we show that for metrics with conormal singularities that correspond to class $C^{1,\alpha}$, $\alpha > 0$, the reflected wave is more regular than the incident wave in a Sobolev sense for a range of background Sobolev spaces. That is, informally, for suitable $s \in \mathbb{R}$ and $\epsilon_0 > 0$, depending on the order of the conormal singularity (thus on α), if a solution of the wave equation is microlocally in the Sobolev space $H^{s-\epsilon_0}_{\mathrm{loc}}$ prior to hitting the conormal singularity of the metric in a normal fashion, then the reflected wave front is in H^s_{loc}, while the transmitted front is just in the a priori space $H^{s-\epsilon_0}_{\mathrm{loc}}$. (This assumes that along the backward continuation of the reflected ray, one has H^s_{loc} regularity, i.e., there is no incident H^s_{loc} singularity for which transmission means propagation along our reflected ray.) Such a result is helpful in the analysis of the multiple scattering series, i.e., for waves iteratively reflecting

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from conormal singularities, since higher order terms, i.e., those involving more reflections, can be effectively 'peeled off' since they have higher regularity.

Here the main interest is in $\alpha < 1$, for in the $C^{1,1}$ setting one has at least a partial understanding of wave propagation *without* a geometric structure to the singularities of the metric, such as conormality (though of course one does need *some* geometric structure to obtain a theorem analogous to ours), as then the Hamilton vector field is Lipschitz, and automatically has unique integral curves; see Smith's paper [17] where a parametrix was constructed, and also the work of Geba and Tataru [2], as well as Taylor's book [21, Chapter 3, Section 11]. We also recall that, in a different direction, for even lower regularity coefficients, Tataru has shown Strichartz estimates [19]-[20]; these are not microlocal in the sense of distinguishing reflected vs. transmitted waves as above. However, the approach of Tataru's work is to allow L^1L^∞ behavior of the second derivatives of the metric, with L^1 behavior being in time, L^∞ in space. Our case has some microlocal similarities, with the L^1 behavior being in a spatial variable (microlocally) normal to the submanifold at which our metric is singular, which should behave similarly for normally incident rays. We also mention that the L^1L^∞ analysis of Tataru's also influenced the work of Smith and Sogge on spectral clusters [18].

In order to state the theorem precisely we need more notation. First suppose X is a $\dim X = n$-dimensional C^∞ manifold, and Y is a smooth embedded submanifold of codimension

$$\operatorname{codim} Y = k.$$

With Hörmander's normalization [7], the class of Lagrangian distributions associated to the conormal bundle N^*Y of Y (also called distributions conormal to Y), denoted by $I^\sigma(N^*Y)$, arises from symbols in $S^{\sigma+(\dim X-2k)/4}$ when parameterized via a partial inverse Fourier transform in the normal variables. That is, if one has local coordinates (x,y), such that Y is given by $x=0$, then $u\in I^\sigma(N^*Y)$ can be written, modulo $C^\infty(\mathbb{R}^n)$, as

$$(2\pi)^{-k}\int e^{ix\cdot\xi}a(y,\xi)\,d\xi,\qquad a\in S^{\sigma+(n-2k)/4}.$$

For us it is sometimes convenient to have the orders relative to delta distributions associated to Y, which arise as the partial inverse Fourier transforms of symbols of order 0, as in [4], thus we let

$$I^{[-s_0]}(Y)=I^{-s_0-(\dim X-2k)/4}(N^*Y),$$

so elements of $I^{[-s_0]}(Y)$ are s_0 orders more regular than such a delta distribution. For any C^∞ vector bundle over X one can then talk about conormal sections (e.g., via local trivialization of the bundle); in particular, one can talk about conormal metrics.

Thus, if X is a C^∞ manifold, Y an embedded submanifold, and g a symmetric 2-cotensor which is in $I^{[-s_0]}(Y)$ with $s_0>k=\operatorname{codim}Y$ (here we drop the bundle from the notation of conormal spaces), then g is continuous. We say that g is Lorentzian if for each $p\in X$, g defines a symmetric bilinear form on T_pX of signature $(1,n-1)$, $n=\dim X$. (One would say g is Riemannian if the signature is $(n,0)$. Another possible normalization of Lorentzian signature is $(n-1,1)$.) We say that Y is time-like if the pull-back of g to Y (which is a C^∞ 2-cotensor) is Lorentzian, or equivalently if the dual metric G restricted to N^*Y is negative definite.

A typical example, with Y time-like, is if $X = X_0 \times \mathbb{R}_t$, where X_0 is the 'spatial' manifold, $Y = Y_0 \times \mathbb{R}$, $g = dt^2 - g_0$, g_0 is (the pull-back of) a Riemannian metric on X_0 which is conormal to Y_0, in the class $I^{[-s_0]}(Y_0)$, where $s_0 > \operatorname{codim}_X Y = \operatorname{codim}_{X_0} Y_0$. In this case, one may choose local coordinates (x, y') on X_0 such that Y_0 is given by $x = 0$; then with $y = (y', t)$, (x, y) are local coordinates on X in which Y is given by $x = 0$. Thus, the time variable t is one of the y variables in this setting.

Before proceeding, recall that there is a propagation of singularities result in the manifolds with corners setting [23], which requires only minimal changes to adapt to the present setting. This states that for solutions of the wave equation lying in $H^{1,r}_{\mathrm{b}}(X)$ for some $r \in \mathbb{R}$, $\mathrm{WF}^{1,m}_{\mathrm{b}}$ propagates along generalized broken bicharacteristics. Thus, for a ray normally incident at Y, if all of the incoming rays that are incident at the same point in Y and that have the same tangential momentum carry H^{m+1} regularity, then the outgoing rays from this point in Y with this tangential momentum will carry the same regularity. In other words, in principle (and indeed, when one has boundaries, or transmission problems with jump singularities of the metric, this is typically the case) H^{m+1} singularities can jump from a ray to another ray incident at the same point with the same tangential momentum (let us call these *related rays*), i.e., one has a whole cone (as the magnitude of the normal momentum is conserved for the rays) of reflected rays carrying the H^{m+1} singularity. Here we recall that for $r \geq 0$, $H^{1,r}_{\mathrm{b}}(X)$ is the subspace of $H^1(X)$ consisting of elements possessing r b (i.e., tangential to Y) derivatives in $H^1(X)$; for $r < 0$ these are distributions obtained from $H^1(X)$ by taking finite linear combinations of up to $-r$ derivatives of elements of $H^1(X)$. In particular, one can have arbitrarily large singularities; one can always represent these by taking tangential derivatives, in particular time derivatives. Via standard functional analytic duality arguments, these estimates (which also hold for the inhomogeneous equation) also give *solvability*, provided there is a global time function t. Phrased in terms of these spaces, and for convenience for the inhomogeneous equation with vanishing initial data, for $f \in H^{-1,r+1}_{\mathrm{b}}(X)$ supported in $t > t_0$ there exists a unique $u \in H^{1,r}_{\mathrm{b}}(X)$ solving the equation $\Box_g u = f$ such that $\operatorname{supp} u \subset \{t > t_0\}$.

The object of this paper is to improve on this propagation result by showing that, when $s_0 > k + 1$ (thus $I^{[-s_0]}(Y) \subset C^{1+\alpha}$ for $\alpha < s_0 - k - 1$) in fact this jump to the related rays does not happen in an appropriate range of Sobolev spaces. As above, let (x, y) denote local coordinates on X, Y given by $x = 0$, and let (ξ, η) denote dual variables. Let $\Sigma \subset T^*X$ denote the characteristic set of the wave operator $\Box = \Box_g$; this is the zero-set of the dual metric G in T^*X.

THEOREM 1.1. – *Suppose* $\operatorname{codim} Y = k = 1$, $k + 1 + 2\epsilon_0 < s_0$ *and* $0 < \epsilon_0 \leq s < s_0 - \epsilon_0 - 1 - k/2$. *Suppose that* $u \in L^2_{\mathrm{loc}}$, $\Box u = 0$,

$$q_0 = (0, y_0, \xi_0, \eta_0) \in \Sigma,\ \xi_0 \neq 0,$$

and the backward bicharacteristics from related *points* $(0, y_0, \xi, \eta_0) \in \Sigma$ *are disjoint from* $\mathrm{WF}^{s-\epsilon_0}(u)$*, and the backward bicharacteristic from the point* q_0 *is disjoint from* $\mathrm{WF}^s(u)$*. Then the forward bicharacteristic from* $(0, y_0, \xi_0, \eta_0)$ *is disjoint from* $\mathrm{WF}^s(u)$*.*

REMARK 1.2. – The theorem is expected to be valid for all values of k, and the limitation on k in the statement is so that it fits conveniently into the existing (b-microlocal) framework