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ELLIPTIC ESTIMATES IN COMPOSITE MEDIA WITH SMOOTH INCLUSIONS: AN INTEGRAL EQUATION APPROACH

BY Habib AMMARI, Eric BONNETIER, Faouzi TRIKI AND Michael VOGELIUS

ABSTRACT. – We consider a scalar elliptic equation for a composite medium consisting of homogeneous $\mathscr{C}^{1,\alpha_0}$ inclusions, $0 < \alpha_0 \leq 1$, embedded in a constant matrix phase. When the inclusions are separated and are separated from the boundary, the solution has an integral representation, in terms of potential functions defined on the boundary of each inclusion. We study the system of integral equations satisfied by these potential functions as the distance between two inclusions tends to 0. We show that the potential functions converge in $\mathscr{C}^{0,\alpha}$, $0 < \alpha < \alpha_0$ to limiting potential functions, with which one can represent the solution when the inclusions are touching. As a consequence, we obtain uniform $\mathscr{C}^{1,\alpha}$ bounds on the solution, which are independent of the inter–inclusion distances.

RÉSUMÉ. – Nous étudions des milieux composites constitués d'inclusions homogènes de forme $\mathscr{C}^{1,\alpha_0}$, immergées dans une phase matrice constante. Lorsque les inclusions ne se touchent pas, la solution de l'équation de diffusion peut être représentée à l'aide de potentiels de surface, solutions d'un système d'équations intégrales. Nous étudions ce système lorsque la distance inter-inclusion tend vers 0. Nous montrons que les potentiels de surface convergent dans $\mathscr{C}^{0,\alpha}$, $0 < \alpha < \alpha_0$, vers des potentiels limites, qui permettent d'obtenir une représentation intégrale du problème limite. Nous en déduisons des estimations sur les solutions dans $\mathscr{C}^{1,\alpha}$, uniformes par rapport à la distance inter-inclusions.

1. Introduction

In a bounded domain $\Omega \subset \mathbb{R}^2$, we consider a composite medium consisting of a finite number of inclusions embedded in a matrix phase. We assume that the inclusions and the matrix have (different) constant, scalar conductivities. The resulting, spatially varying, piecewise constant conductivity is denoted by $a(\cdot)$. Given a current g on the boundary $\partial\Omega$, with $\int_{\partial\Omega} g \, d\sigma = 0$, we consider the solution u to the elliptic equation

$$\nabla \cdot (a(\cdot)\nabla u) = 0 \quad \text{in } \Omega, \quad \text{with} \quad a(\cdot)\partial_\nu u = g \quad \text{on } \partial\Omega,$$

in other words, we consider the continuous function u, which is harmonic in each inclusion as well as in the matrix, which satisfies the usual transmission conditions across the inclusion

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boundaries, and which has the prescribed co-normal derivative g on $\partial\Omega$. To make u unique we impose the condition $\int_{\partial\Omega} u \, d\sigma = 0$.

In this paper, we are interested in a priori estimates for the solution u, and in particular its gradient. We assume that Ω has a smooth boundary, and that the imposed current is smooth. When the inclusions are merely Lipschitz, it is well known (from elliptic theory in domains with corners) that ∇u is generally not uniformly bounded, i.e., generally not in L^∞. On the other hand, when the inclusions are smooth (say $\mathscr{C}^{1,\alpha_0}, 0 < \alpha_0 \leq 1$) and when they are not mutually touching and do not touch $\partial\Omega$, it is equally well known that ∇u is bounded. A natural question is whether ∇u stays uniformly bounded, even as some of the inclusions get close.

This question has been addressed in several papers (see for example [11] and [18]). It has been established that ∇u is bounded in $L^\infty(\Omega)$ independly of the distance between the smooth inclusions. The answer given in [18] is actually quite a bit more general. It addresses the case of a divergence form elliptic equation with 'piecewise Hölder coefficients': assume there exist numbers $0 < \alpha_0, 0 < c_0, \mu \leq 1, 0 < \lambda_0 < \Lambda_0$, and a positive integer M such that

i. Ω contains M possibly touching inclusions D_l, $1 \leq l \leq M$, each of which is a $\mathscr{C}^{1,\alpha_0}$ subdomain.
ii. For any $1 \leq l \leq M$, $\operatorname{dist}(\overline{D_l}, \partial\Omega) > c_0 > 0$,
iii. In each inclusion, and in the remaining part $D_{M+1} := \Omega \backslash \cup_{1 \leq l \leq M} \overline{D_l}$, the conductivity satisfies $\lambda_0 < a_{|\overline{D}_l} < \Lambda_0$, and has $\mathscr{C}^\mu$ regularity.

Then

$$(1) \qquad \sum_{l=1}^{M+1} ||u||_{\mathscr{C}^{1,\alpha}(\overline{D}_l \cap \Omega_\varepsilon)} \leq C ||g||_{L^2(\partial\Omega)}, \quad \text{for any } 0 < \alpha < \min\{\mu, \frac{\alpha_0}{2(\alpha_0+1)}\},$$

where Ω_ε, $\varepsilon > 0$, denotes the set

$$\Omega_\varepsilon = \{X \in \Omega, \ \operatorname{dist}(X, \partial\Omega) > \varepsilon\}.$$

The constant C depends on $\varepsilon, \alpha, M, \lambda_0, \Lambda_0, \mu, \Omega$ and the appropriate $\mathscr{C}^{1,\alpha}$ "norms" of the parametrizations of the inclusion boundaries. But note that C is independent of the inter–inclusion distance. The proof given in [18] uses elliptic blow-up techniques and maximum principles, and is thus restricted to scalar problems.

In a subsequent paper [19], Y.-Y. Li and L. Nirenberg extended the above result to strongly elliptic systems, with the same restriction $0 < \alpha < \min\{\mu, \frac{\alpha_0}{2(\alpha_0+1)}\}$ for the regularity "measure" of u. Recently, G. Citti and F. Ferrari [13] followed the approach of [18], using more precise estimates and obtained an improved *regularity* result. They show that the solution u is locally in $\mathscr{C}^{1,\alpha}$, for $\alpha \leq \min\{\mu, \alpha_0\}$. However, they assume that the inclusions are strictly separated from one another, and their proof yields regularity estimates that may depend on the inter-inclusion distance. The uniform character of the estimates is the cardinal point of [18] and of our work.

In the case of perfectly conducting or perfectly insulating inclusions the gradients may blow up as the inter-inclusion distance, δ, approaches 0. The estimates (1) are therefore not uniform in the magnitude of the conductivities. In [7], the solution for perfectly conducting inclusions is shown to satisfy

$$\begin{cases} ||\nabla u||_{L^\infty} \leq \frac{C}{\sqrt{\delta}}||u||_{L^2(\partial\Omega)} & \text{for } n=2, \\ ||\nabla u||_{L^\infty} \leq \frac{C}{\delta|\ln\delta|}||u||_{L^2(\partial\Omega)} & \text{for } n=3, \\ ||\nabla u||_{L^\infty} \leq \frac{C}{\delta}||u||_{L^2(\partial\Omega)} & \text{for } n=4, \end{cases} \tag{2}$$

where n is the ambient dimension. The case $n = 2$ was derived independently by Yun, using conformal mapping techniques [22]. The picture is less complete for the case of insulating inclusions, see [8].

For $n = 2$ and for circular inclusions, one can obtain very precise bounds in terms of both contrast and inter-inclusion distance, since the solution has a series representation that lends itself to asymptotic analysis [5, 3, 12, 20]. Optimal upper and lower bounds on the potential gradients are derived in [5, 3] for nearly touching pairs of circular inclusions. In the case of two disks, a decomposition of the solution into a singular part, and a part that remains uniformly bounded with respect to δ, is given in [4].

When the conductivity is piecewise constant, and when the inclusions are $\mathscr{C}^{1,\alpha_0}$, mutually separated and separated from the boundary, then one can represent u in the form

$$u(X) = \sum_{l=1}^{M} S_l\varphi_l(X) + H(X), \tag{3}$$

where H is a harmonic function, where each φ_l is defined on ∂D_l, and where S_l denotes the single layer potential on ∂D_l. Invoking the transmission conditions on ∂D_l and the Neumann condition on $\partial\Omega$, we can derive a system of integral equations, for the φ_l's, and an associated (implicit) formula for H. As each inclusion has $\mathscr{C}^{1,\alpha_0}$ regularity, results from classical potential theory (see, e.g., [15]) show that this system is invertible. Detailed facts about the regularity of u may be deduced from the representation (3).

The aim of this paper is to show that the system of integral equations for the φ_l's is uniformly invertible in $\mathscr{C}^{0,\alpha}$ as inclusions get close. The associated uniform estimates on the inverse can then be used to derive a priori estimates for the solution, u, in $\mathscr{C}^{1,\alpha}$ norms.

The integral representation (3) of solutions has also been used in other related contexts. In particular, recent works have focused on the connection between the bounds on ∇u and the spectral properties of the kernel of the integral equation system (the Neumann–Poincaré operator) for varying coefficient contrast and inter-inclusion distance [1, 10].

For simplicity we always assume that the inclusions are convex, and that any two that asymptotically meet only meet at one point. Since the regularity of u and the corresponding estimates only depend on the geometry of the inclusions locally, we shall restrict ourselves to the case of two inclusions, D_1 and D_2, of size $O(1)$, that asymptotically meet (with a horizontal tangent) at the point 0, see Figure 1. We denote $\Gamma_i := \partial D_i, i = 1, 2$. For simplicity, we assume that the matrix phase has conductivity 1 and that both inclusions have conductivity $k \neq 1$. For $\delta > 0$, we consider the situation where the inclusions are at a distance δ apart, say in the unit vertical direction e_2. As we shall see, the corresponding system of