ISSN 0012-9593

quatrième série - tome 48 fascicule 3 mai-juin 2015

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Dan COMAN & George MARINESCU

Equidistribution results for singular metrics on line bundles

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
annales@ens.fr

Édition / *Publication*

Société Mathématique de France
Institut Henri Poincaré
11, rue Pierre et Marie Curie
75231 Paris Cedex 05
Tél. : (33) 01 44 27 67 99
Fax : (33) 01 40 46 90 96

Abonnements / *Subscriptions*

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 09
Fax : (33) 04 91 41 17 51
email : smf@smf.univ-mrs.fr

Tarifs

Europe : 515 €. Hors Europe : 545 €. Vente au numéro : 77 €.

ISSN 0012-9593

Directeur de la publication : Marc Peigné
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4ᵉ série, t. 48, 2015, p. 497 à 536

EQUIDISTRIBUTION RESULTS FOR SINGULAR METRICS ON LINE BUNDLES

BY DAN COMAN AND GEORGE MARINESCU

ABSTRACT. – Let (L,h) be a holomorphic line bundle with a positively curved singular Hermitian metric over a complex manifold X. One can define naturally the sequence of Fubini-Study currents γ_p associated to the space of L^2-holomorphic sections of $L^{\otimes p}$. Assuming that the singular set of the metric is contained in a compact analytic subset Σ of X and that the logarithm of the Bergman density function of $L^{\otimes p}|_{X\setminus\Sigma}$ grows like $o(p)$ as $p\to\infty$, we prove the following:

1) the currents γ_p^k converge weakly on the whole X to $c_1(L,h)^k$, where $c_1(L,h)$ is the curvature current of h.

2) the expectations of the common zeros of a random k-tuple of L^2-holomorphic sections converge weakly in the sense of currents to $c_1(L,h)^k$.

Here k is so that $\operatorname{codim}\Sigma\geq k$. Our weak asymptotic condition on the Bergman density function is known to hold in many cases, as it is a consequence of its asymptotic expansion. We also prove it here in a quite general setting. We then show that many important geometric situations (singular metrics on big line bundles, Kähler-Einstein metrics on Zariski-open sets, arithmetic quotients) fit into our framework.

RÉSUMÉ. – Considérons un fibré holomorphe en droites L muni d'une métrique singulière h au-dessus d'une variété complexe X. Soit γ_p le courant de Fubini-Study associé naturellement à l'espace des sections holomorphes de carré intégrable de $L^{\otimes p}$. En supposant que le lieu singulier de la métrique h est contenu dans un ensemble analytique compact $\Sigma\subset X$ tel que $\operatorname{codim}\Sigma\geq k$ et que le logarithme du noyau de Bergman associé à $L^{\otimes p}|_{X\setminus\Sigma}$ a l'ordre de croissance $o(p)$, $p\to\infty$, nous prouvons que :

1) Les courants γ_p^k convergent faiblement sur X vers $c_1(L,h)^k$, où $c_1(L,h)$ est le courant de courbure de h.

2) Les moyennes des zéros communs d'un k-vecteur aléatoire de sections holomphes L^2-intégrables convergent faiblement dans le sens des courants vers $c_1(L,h)^k$.

D. Coman was partially supported by the NSF Grants DMS-0900934 and DMS-1300157. G. Marinescu was partially supported by DFG funded projects SFB/TR 12, MA 2469/2-1.

L'hypothèse de croissance du noyau de Bergman est la conséquence de son développement asymptotique dans le cas d'une métrique lisse h. Nous la démontrons ici sous des conditions assez générales. Nous montrons ensuite que nos résultats s'appliquent à nombre de situations géométriques (métriques singulières sur un fibré gros, métriques de Kähler-Einstein sur des ouverts de Zariski, quotients arithmétiques...).

1. Introduction

Let X be a compact complex manifold of dimension n, $L \longrightarrow X$ be a positive holomorphic line bundle, and h be a smooth Hermitian metric on L whose curvature $c_1(L,h)$ is a positive (1,1) form on X. Let $\Phi_p : X \longrightarrow \mathbb{P}^{d_p-1}$ be the Kodaira map defined by an orthonormal basis of $H^0(X,L^p)$ with respect to the inner product given by the metric induced by h on $L^p := L^{\otimes p}$ and a fixed volume form on X, where $d_p = \dim H^0(X,L^p)$. The pull-back $\Phi_p^\star(\omega_{FS})$ of the Fubini-Study form ω_{FS} is a smooth (1,1) form for all p sufficiently large, since Φ_p is an embedding by Kodaira's embedding theorem. A theorem of Tian [59] (with improvements by Ruan [50]) asserts that $\frac{1}{p}\,\Phi_p^\star(\omega_{FS}) \to c_1(L,h)$ as $p \to \infty$, in the $\mathscr{C}^\infty$ topology on X.

Tian's theorem is a consequence of the asymptotic expansion of the Bergman density function associated to the inner product on $H^0(X,L^p)$ mentioned above. In the context of positive line bundles this asymptotic expansion is proved in various forms in [59, 11, 62, 17, 42, 43, 44, 6]. For line bundles endowed with arbitrary smooth Hermitian metrics the Bergman density function behavior and important consequences are studied in [5] and [7].

In the case of holomorphic Hermitian line bundles over complete Hermitian manifolds the asymptotic expansion of the Bergman density function associated to the corresponding spaces of L^2-holomorphic sections was proved in [44] (see also [42, 43]). In particular, a version of Tian's theorem was obtained for a big line bundle L over a (compact) manifold X. Such a line bundle admits a singular Hermitian metric h, smooth outside a proper analytic subvariety $\Sigma \subset X$, and whose curvature current $c_1(L,h)$ is strictly positive. It is shown in [43, Section 6.2] that there exist a smooth positively curved Hermitian metric h_ε on $L|_{X\setminus\Sigma}$, which is a small perturbation of h, and a smooth positive (1,1) form Θ defining a generalized Poincaré metric on $X \setminus \Sigma$, so that the following hold. If $H^0_{(2)}(X\setminus\Sigma,L^p)$ is the space of L^2-holomorphic sections of $L^p|_{X\setminus\Sigma}$ relative to the metrics h_ε and Θ then $H^0_{(2)}(X\setminus\Sigma,L^p) \subset H^0(X,L^p)$, so a Kodaira map $\Phi_p : X \dashrightarrow \mathbb{P}^{d_p-1}$ can be defined by using an orthonormal basis of $H^0_{(2)}(X\setminus\Sigma,L^p)$. Let $\gamma_p = \Phi_p^\star(\omega_{FS})$ and $\omega = c_1(L|_{X\setminus\Sigma},h_\varepsilon)$. Then $\frac{1}{p}\,\gamma_p \to \omega$ as $p \to \infty$, locally uniformly in the $\mathscr{C}^\infty$ topology on $X \setminus \Sigma$.

Since γ_p are currents on X it is natural to try and study the weak convergence of the sequence $\{\gamma_p/p\}$, and to ask whether a global version of Tian's theorem holds in this setting. We will show that this is indeed the case.

Let us work in the following more general setting:

(A) X is a complex manifold of dimension n (not necessarily compact), Σ is a compact analytic subvariety of X, and Ω is a smooth positive $(1,1)$ form on X.

(B) (L, h) is a holomorphic line bundle on X with a singular (semi)positively curved Hermitian metric h which is continuous on $X \setminus \Sigma$. We denote by h_p the Hermitian metric induced by h on $L^p := L^{\otimes p}$.

(C) The volume form on $X \setminus \Sigma$ is $f\Omega^n$, where $f \in L^1_{\mathrm{loc}}(X \setminus \Sigma, \Omega^n)$ verifies $f \geq c_x > 0$ Ω^n-a.e. in a neighborhood U_x of each $x \in (X \setminus \Sigma) \cup \Sigma^{n-1}_{\mathrm{reg}}$. Here $\Sigma^{n-1}_{\mathrm{reg}}$ is the set of regular points y where $\dim_y \Sigma = n-1$.

We denote the curvature current of h by $\gamma = c_1(L, h)$ and consider the space $H^0_{(2)}(X \setminus \Sigma, L^p)$ of L^2-holomorphic sections of $L^p|_{X\setminus\Sigma}$ relative to the metric h_p on L^p and the volume form $f\Omega^n$ on $X \setminus \Sigma$, endowed with the inner product

$$(S, S')_p = \int_{X\setminus\Sigma} \langle S, S' \rangle_{h_p} f\Omega^n\,, \quad \text{where } \langle S, S' \rangle_{h_p} = h_p(S, S'),\ S, S' \in H^0_{(2)}(X \setminus \Sigma, L^p).$$

We let $\|S\|^2_p = (S, S)_p$. Since $H^0_{(2)}(X \setminus \Sigma, L^p)$ is separable, let $\{S^p_j\}_{j\geq 1}$ be an orthonormal basis and denote by P_p the Bergman density function defined by

$$P_p(x) = \sum_{j=1}^{\infty} |S^p_j(x)|^2_{h_p},\quad |S^p_j(x)|^2_{h_p} := \langle S^p_j(x), S^p_j(x) \rangle_{h_p},\ x \in X \setminus \Sigma. \tag{1}$$

Note that this definition is independent of the choice of basis, and the function P_p is continuous on $X \setminus \Sigma$ (see Section 3).

Next we define the Fubini-Study currents γ_p on $X \setminus \Sigma$ by

$$\gamma_p|_U = \frac{1}{2}\, dd^c \log \left(\sum_{j=1}^{\infty} |s^p_j|^2 \right),\ U \subset X \setminus \Sigma \text{ open}\,, \tag{2}$$

where $d^c = \frac{1}{2\pi i}(\partial - \overline{\partial})$, $S^p_j = s^p_j e^{\otimes p}$, and e is a local holomorphic frame for L on U.

One of our main results is the following:

Theorem 1.1. – *If $X, \Sigma, (L, h), f, \Omega$ verify assumptions* (A)-(C) *then $H^0_{(2)}(X \setminus \Sigma, L^p) \subset H^0(X, L^p)$ and γ_p extends to a positive closed current on X defined locally by Formula* (2) *and which is independent of the choice of basis $\{S^p_j\}_{j\geq 1}$. Assume further that*

$$\lim_{p\to\infty} \frac{1}{p} \log P_p(x) = 0,\ \textit{locally uniformly on } X \setminus \Sigma. \tag{3}$$

Then $\frac{1}{p}\gamma_p \to \gamma$ weakly on X. If, in addition, $\dim \Sigma \leq n-k$ for some $2 \leq k \leq n$, then the currents γ^k and γ^k_p are well defined on X, respectively on each relatively compact neighborhood of Σ, for all p sufficiently large. Moreover, $\frac{1}{p^k}\gamma^k_p \to \gamma^k$ weakly on X.

This theorem is proved in Section 3. The proof relies on a local continuity property of the complex Monge-Ampère operator which is of independent interest (see Theorem 3.4). Some background material about singular Hermitian metrics and pluripotential theory needed in the paper is recalled in Section 2. We note here that if $\operatorname{codim} \Sigma < k$ the current γ^k cannot be defined (see [9, 10]), so the assumption on the dimension in Theorem 1.1 is optimal.

We examine in Section 6 a series of important situations where condition (3) of Theorem 1.1 holds, as it is an immediate consequence of deep results regarding the asymptotic expansion of the Bergman density function $P_p(x) \sim b_0(x)p^n + b_1(x)p^{n-1} + \cdots$. Especially, Theorem 1.1 yields equidistribution results for singular metrics on big line bundles