ISSN 0012-9593

quatrième série - tome 48 fascicule 3 mai-juin 2015

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Ken ABE & Yoshikazu GIGA & Matthias HIEBER

Stokes Resolvent Estimates in Spaces of Bounded Functions

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
annales@ens.fr

Édition / *Publication*

Société Mathématique de France
Institut Henri Poincaré
11, rue Pierre et Marie Curie
75231 Paris Cedex 05
Tél. : (33) 01 44 27 67 99
Fax : (33) 01 40 46 90 96

Abonnements / *Subscriptions*

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 09
Fax : (33) 04 91 41 17 51
email : smf@smf.univ-mrs.fr

Tarifs

Europe : 515 €. Hors Europe : 545 €. Vente au numéro : 77 €.

ISSN 0012-9593

Directeur de la publication : Marc Peigné
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4 e série, t. 48, 2015, p. 537 à 559

STOKES RESOLVENT ESTIMATES IN SPACES OF BOUNDED FUNCTIONS

BY KEN ABE, YOSHIKAZU GIGA AND MATTHIAS HIEBER

ABSTRACT. – The Stokes equation on a domain $\Omega \subset \mathbf{R}^n$ is well understood in the L^p-setting for a large class of domains including bounded and exterior domains with smooth boundaries provided $1 < p < \infty$. The situation is very different for the case $p = \infty$ since in this case the Helmholtz projection does not act as a bounded operator anymore. Nevertheless it was recently proved by the first and the second author of this paper by a contradiction argument that the Stokes operator generates an analytic semigroup on spaces of bounded functions for a large class of domains. This paper presents a new approach as well as new a priori L^∞-type estimates to the Stokes equation. They imply in particular that the Stokes operator generates a C_0-analytic semigroup of angle $\pi/2$ on $C_{0,\sigma}(\Omega)$, or a non-C_0-analytic semigroup on $L^\infty_\sigma(\Omega)$ for a large class of domains. The approach presented is inspired by the so called Masuda-Stewart technique for elliptic operators. It is shown furthermore that the method presented applies also to different types of boundary conditions as, e.g., Robin boundary conditions.

RÉSUMÉ. – L'équation de Stokes sur un ouvert $\Omega \subset \mathbf{R}^n$ a été bien étudiée dans le cadre de L^p pour $1 < p < \infty$ et pour une grande classe d'ouverts réguliers. La situation est bien différente pour le cas $p = \infty$, car la projection de Leray n'est pas bornée dans ce cas. Il a été démontré par les premier et second auteurs de cet article que l'opérateur de Stokes engendre tout de même un semigroupe holomorphe sur des espaces de fonctions bornées pour une grande classe d'ouverts. Cet article présente une nouvelle approche et des nouvelles estimations *a priori* de type L^∞ pour l'équation de Stokes. Celles-ci impliquent en particulier que l'opérateur de Stokes engendre un semigroupe holomorphe d'angle $\pi/2$ sur $L^\infty_\sigma(\Omega)$ (pas fortement continu) ou $C_{0,\sigma}(\Omega)$ pour une grande classe d'ouverts Ω. L'approche est inspirée par la méthode de Masuda-Stewart. D'autre part, il est démontré que la méthode s'applique aussi à d'autres conditions de bord, par exemple aux conditions de Robin.

1. Introduction and main results

The investigation of the linear Stokes equations as well as properties and corresponding estimates are often basis for the analysis of the nonlinear Navier-Stokes equations. In particular, analyticity of the solution operator (called the Stokes semigroup) plays a fundamental role for studying the Navier-Stokes equations. It is well-known that the Stokes semigroup forms an analytic semigroup on $L^p_\sigma(\Omega)$ for $p \in (1,\infty)$, the space of L^p-solenoidal vector fields, for various kind of domains $\Omega \subset \mathbf{R}^n, n \geq 2$ including bounded and exterior domains having smooth boundaries; see, e.g., [37, 20]. By now, analyticity results are known for other type unbounded domains, see [17, 18], [4] ([6, 5] with variable viscosity coefficients) and Lipschitz domains [33]. An $\tilde{L}^p$-theory is developed in [13, 14], [15] for a general domain. Moreover, L^p-theory is investigated in [19] for unbounded domains, for which the Helmholtz projection is bounded.

It is the aim of this paper to consider the case $p = \infty$. Note that the Helmholtz projection is no longer bounded in L^∞ even if $\Omega = \mathbf{R}^n$. When $\Omega = \mathbf{R}^n_+$, the analyticity of the semigroup is known in L^∞-type spaces including $C_{0,\sigma}(\Omega)$, the L^∞-closure of $C^\infty_{c,\sigma}(\Omega)$, the space of all smooth solenoidal vector fields compactly supported in Ω [11] (see also [38, 28]). Their approach is based on explicit calculations of the solution operator $R(\lambda) : f \mapsto v = v_\lambda$ of the corresponding resolvent problem of

$$\lambda v - \Delta v + \nabla q = f \quad \text{in } \Omega, \tag{1.1}$$

$$\operatorname{div} v = 0 \quad \text{in } \Omega, \tag{1.2}$$

$$v = 0 \quad \text{on } \partial\Omega. \tag{1.3}$$

As recently shown in [2, 3] by a blow-up argument to the non-stationary Stokes equations, it turns out that the Stokes semigroup is extendable to an analytic semigroup on $C_{0,\sigma}$ for what is called *admissible domains* which include bounded and exterior domains having boundaries of class C^3.

In this paper, we present a direct resolvent approach to the Stokes resolvent equations (1.1)–(1.3) and establish the a priori estimate of the form

$$\begin{aligned} &M_p(v,q)(x,\lambda) \\ &= |\lambda||v(x)| + |\lambda|^{1/2}|\nabla v(x)| + |\lambda|^{n/2p}\|\nabla^2 v\|_{L^p(\Omega_{x,|\lambda|^{-1/2}})} + |\lambda|^{n/2p}\|\nabla q\|_{L^p(\Omega_{x,|\lambda|^{-1/2}})}, \end{aligned}$$

for $p > n$ and

$$\sup_{\lambda \in \Sigma_{\vartheta,\delta}} \|M_p(v,q)\|_{L^\infty(\Omega)}(\lambda) \leq C\|f\|_{L^\infty(\Omega)} \tag{1.4}$$

for some constant $C > 0$ independent of f. Here, $\Omega_{x,r}$ denotes the intersection of Ω with an open ball $B_x(r)$ centered at $x \in \Omega$ with radius $r > 0$, i.e., $\Omega_{x,r} = B_x(r) \cap \Omega$ and $\Sigma_{\vartheta,\delta}$ denotes the sectorial region in the complex plane given by $\Sigma_{\vartheta,\delta} = \{\lambda \in \mathbf{C}\backslash\{0\} \mid |\arg\lambda| < \vartheta,\ |\lambda| > \delta\}$ for $\vartheta \in (\pi/2, \pi)$ and $\delta > 0$. Our approach is inspired by the corresponding approach for general elliptic operators. K. Masuda was the first to prove analyticity of the semigroup associated to general elliptic operators in $C_0(\mathbf{R}^n)$ including the case of higher orders [29, 31] ([30]). This result was then extended by H. B. Stewart to the case for the Dirichlet problem [39] and more general boundary condition [40]. This Masuda-Stewart method was applied to many other situations [7, 26], [22, 8], [9]. However, its application to the resolvent Stokes equations (1.1)–(1.3) was unknown.

In the sequel, we prove the estimate (1.4) by using the L^p-estimates for the Stokes resolvent equations with inhomogeneous divergence condition [16, 17]. We invoke *strictly admissibility* of a domain introduced in [3, Definition 2.4] which implies an estimate of pressure q in terms of the velocity by

$$\sup_{x\in\Omega} d_\Omega(x)|\nabla q(x)| \le C_\Omega \|W(v)\|_{L^\infty(\partial\Omega)} \quad \text{for } W(v) = -(\nabla v - \nabla^T v)n_\Omega, \tag{1.5}$$

where ∇f denotes $(\partial f_i/\partial x_j)_{1\le i,j\le n}$ and $\nabla^T f = (\nabla f)^T$ for a vector field $f = (f_i)_{1\le i\le n}$. The estimate (1.5) plays a key role in transferring results from the elliptic situation to the situation of the Stokes system. Here, n_Ω denotes the unit outward normal vector field on $\partial\Omega$ and d_Ω denotes the distance function from the boundary, i.e., $d_\Omega(x) = \inf_{y\in\partial\Omega}|x-y|$ for $x \in \Omega$. The estimate (1.5) can be viewed as a regularizing-type estimate for solutions to the Laplace equation $\Delta P = 0$ in Ω with the Neumann boundary condition $\partial P/\partial n_\Omega = \mathrm{div}_{\partial\Omega} W$ on $\partial\Omega$ for a tangential vector field W, where $\mathrm{div}_{\partial\Omega} = \mathrm{tr}\,\nabla_{\partial\Omega}$ denotes the surface divergence and $\nabla_{\partial\Omega} = \nabla - n_\Omega(n_\Omega\cdot\nabla)$ is the gradient on $\partial\Omega$. It is known that $P = q$ solves this Neumann problem for $W = W(v)$ given by (1.5) [3, Lemma 2.8] and the estimate (1.5) holds for bounded domains [2] and exterior domains [3]. Note that when $n = 3$, $W(v)$ is nothing but a tangential trace of vorticity, i.e., $W(v) = -\mathrm{curl}\, v \times n_\Omega$. We call Ω *strictly admissible* if there exists a constant $C = C_\Omega$ such that the a priori estimate

$$\|\nabla P\|_{L^\infty_d(\Omega)} \le C\|W\|_{L^\infty(\partial\Omega)} \tag{1.6}$$

holds for all solutions P of the Neumann problem for a tangential vector field $W \in L^\infty(\partial\Omega)$. Here $L^\infty_d(\Omega)$ denotes the space of all locally integrable functions f such that $d_\Omega f$ is essentially bounded in Ω. The space $L^\infty_d(\Omega)$ is equipped with the norm $\|f\|_{L^\infty_d(\Omega)} = \sup_{x\in\Omega} d_\Omega(x)|f(x)|$. The meaning of a solution is understood in the weak sense, i.e., we say $\nabla P \in L^\infty_d(\Omega)$ is a solution for the Neumann problem if $\int_\Omega P\Delta\varphi dx = \int_{\partial\Omega} W\cdot \nabla_{\partial\Omega}\varphi d\mathcal{H}^{n-1}(x)$ holds for all $\varphi \in C^2_c(\bar\Omega)$ satisfying $\partial\varphi/\partial n_\Omega = 0$ on $\partial\Omega$, where $\mathcal{H}^{n-1}$ denotes the $n-1$-dimensional Hausdorff measure; see also [3, Definition 2.3].

We are now in the position to formulate the main results of this paper.

THEOREM 1.1. – *Let Ω be a strictly admissible, uniformly C^2-domain in $\mathbf{R}^n$, $n \ge 2$. Let $p > n$. For $\vartheta \in (\pi/2, \pi)$, there exist constants δ and C such that the a priori estimate (1.4) holds for all solutions $(v, \nabla q) \in W^{2,p}_{\mathrm{loc}}(\bar\Omega)\times(L^p_{\mathrm{loc}}(\bar\Omega)\cap L^\infty_d(\Omega))$ of (1.1)–(1.3) for $f \in C_{0,\sigma}(\Omega)$ and $\lambda \in \Sigma_{\vartheta,\delta}$.*

The a priori estimate (1.4) implies the analyticity of the Stokes semigroup in L^∞-type spaces. Let us observe the generation of an analytic semigroup in $C_{0,\sigma}(\Omega)$. By the $\tilde{L}^p$-theory [13, 14], [15] we verify the existence of a solution to (1.1)–(1.3), $(v, \nabla q) \in W^{2,p}_{\mathrm{loc}}(\bar\Omega) \times (L^p_{\mathrm{loc}}(\bar\Omega) \cap L^\infty_d(\Omega))$ for $f \in C^\infty_{c,\sigma}(\Omega)$ in a uniformly C^2-domain Ω. The solution operator $R(\lambda)$ is then uniquely extendable to $C_{0,\sigma}(\Omega)$ by the uniform approximation together with the estimate (1.4). Here, the solution operator to the pressure gradient $f \mapsto \nabla q_\lambda$ is also uniquely extended for $f \in C_{0,\sigma}$. We observe that $R(\lambda)$ is injective on $C_{0,\sigma}$ since the estimate (1.5) immediately implies that $f = 0$ for $f \in C_{0,\sigma}$ such that $v_\lambda = R(\lambda)f = 0$. The operator $R(\lambda)$ may be regarded as a surjective operator from $C_{0,\sigma}$ to the range of $R(\lambda)$. The open mapping theorem then implies the existence of a closed operator A such that $R(\lambda) = (\lambda - A)^{-1}$; see [10, Proposition B.6]. We call *A the Stokes operator* in $C_{0,\sigma}(\Omega)$. From Theorem 1.1, we obtain: