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NONLINEAR ANALYSIS WITH RESURGENT FUNCTIONS

BY David SAUZIN

ABSTRACT. – We provide estimates for the convolution product of an arbitrary number of "resurgent functions", that is holomorphic germs at the origin of $\mathbb{C}$ that admit analytic continuation outside a closed discrete subset of $\mathbb{C}$ which is stable under addition. Such estimates are then used to perform nonlinear operations like substitution in a convergent series, composition or functional inversion with resurgent functions, and to justify the rules of "alien calculus"; they also yield implicitly defined resurgent functions. The same nonlinear operations can be performed in the framework of Borel-Laplace summability.

RÉSUMÉ. – Nous obtenons des estimations pour le produit de convolution d'un nombre arbitraire de « fonctions résurgentes », c'est-à-dire de fonctions holomorphes à l'origine du plan complexe qui se prolongent anaytiquement en dehors d'un sous-ensemble fermé discret stable par addition. Ces estimations sont ensuite utilisées pour effectuer des opérations non linéaires avec les fonctions résurgentes, comme la substitution dans une série convergente, la composition ou l'inversion fonctionnelle, et pour justifier les règles du « calcul étranger » ; elles permettent aussi d'obtenir un théorème des fonctions résurgentes implicites. Les mêmes opérations non linéaires peuvent être effectuées dans le cadre de la sommabilité de Borel-Laplace.

1. Introduction

In the 1980s, to deal with local analytic problems of classification of dynamical systems, J. Écalle started to develop his theory of resurgent functions and alien derivatives [10, 11], [13], which is an efficient tool for dealing with divergent series arising from complex dynamical systems or WKB expansions, analytic invariants of differential or difference equations, linear and nonlinear Stokes phenomena [25, 26], [12, 3], [6, 2], [7], [20, 29], [34, 4], [35, 22], [24, 17], [32, 21], [8, 9]; connections were also recently found with Painlevé asymptotics [19], Quantum Topology [18, 5] and Wall Crossing [23].

The starting point in Écalle's theory is the definition of certain subalgebras of the algebra of formal power series by means of the formal Borel transform

$$\mathcal{B}\colon\ \tilde\varphi(z)=\sum_{n=0}^{\infty}a_n z^{-n-1}\in z^{-1}\mathbb{C}[[z^{-1}]]\mapsto\hat\varphi(\zeta)=\sum_{n=0}^{\infty}a_n\frac{\zeta^n}{n!}\in\mathbb{C}[[\zeta]] \tag{1}$$

(using negative power expansions in the left-hand side and changing the name of the indeterminate from z to ζ are just convenient conventions).

It turns out that, for a lot of interesting functional equations, one can find formal solutions which are divergent for all z and whose Borel transforms define holomorphic germs at 0 with particular properties of analytic continuation. The simplest examples are the Euler series [3, 32], which can be written $\tilde\varphi^{\mathrm{E}}(z)=\sum_{n=0}^{\infty}(-1)^n n! z^{-n-1}$ and solves a first-order linear non-homogeneous differential equation, and the Stirling series [10, Vol. 3]

$$\tilde\varphi^{\mathrm{S}}(z)=\sum_{k=1}^{\infty}\frac{B_{2k}}{2k(2k-1)}z^{-2k+1}$$

(here expressed in terms of the Bernoulli numbers), solution of a linear non-homogeneous difference equation derived from the functional equation for Euler's Gamma function by taking logarithms. In both examples the Borel transform gives rise to convergent series with a meromorphic extension to the ζ-plane, namely $(1+\zeta)^{-1}$ for the Euler series and $\zeta^{-2}\left(\frac{\zeta}{2}\coth\frac{\zeta}{2}-1\right)$ for the Stirling series (see [37]). In fact, holomorphic germs at 0 with meromorphic or algebraic analytic continuation are examples of "resurgent functions"; more generally, what is required for a resurgent function is the possibility of following the analytic continuation without encountering natural barriers.

One is thus led to distinguish certain subspaces $\hat{\mathscr{R}}$ of $\mathbb{C}\{\zeta\}$, characterized by properties of analytic continuation which ensure a locally discrete set of singularities for each of its members (and which do not preclude multiple-valuedness of the analytic continuation), and to consider

$$\tilde{\mathscr{R}}\coloneqq\mathbb{C}\oplus\mathcal{B}^{-1}(\hat{\mathscr{R}})\subset\mathbb{C}[[z^{-1}]].$$

Typically one has the strict inclusion $\mathbb{C}\{z^{-1}\}\subsetneq\tilde{\mathscr{R}}$ but the divergent series in $\tilde{\mathscr{R}}$ can be "summed" by means of Borel-Laplace summation. The formal series in $\tilde{\mathscr{R}}$ as well as the holomorphic functions whose germ at 0 belongs to $\hat{\mathscr{R}}$ are termed "resurgent". (One also defines, for each $\omega\in\mathbb{C}^*$, an "alien operator" which measures the singularities at ω of certain branches of the analytic continuation of $\hat\varphi$.)

Later we shall be more specific about the definition of $\hat{\mathscr{R}}$. This article is concerned with the convolution of resurgent functions: the convolution in $\mathbb{C}\{\zeta\}$ is the commutative associative product defined by

$$\hat\varphi_1*\hat\varphi_2(\zeta)=\int_0^{\zeta}\hat\varphi_1(\zeta_1)\hat\varphi_2(\zeta-\zeta_1)\,\mathrm{d}\zeta_1\qquad\text{for } |\zeta| \text{ small enough,} \tag{2}$$

for any $\hat\varphi_1,\hat\varphi_2\in\mathbb{C}\{\zeta\}$, which reflects the Cauchy product of formal series via the formal Borel transform:

$$\mathcal{B}\tilde\varphi_1=\hat\varphi_1\ \text{ and }\ \mathcal{B}\tilde\varphi_2=\hat\varphi_2\quad\Longrightarrow\quad\mathcal{B}(\tilde\varphi_1\tilde\varphi_2)=\hat\varphi_1*\hat\varphi_2.$$

Since the theory was designed to deal with nonlinear problems, it is of fundamental importance to control the convolution product of resurgent functions; however, this requires to follow the analytic continuation of the function defined by (2), which turns out not to be an easy task. In fact, probably the greatest difficulties in understanding and applying resurgence theory are connected with the problem of controlling the analytic continuation of functions defined by such integrals or by analogous multiple integrals. Even the mere stability under convolution of the spaces $\hat{\mathscr{R}}$ is not obvious [10, 3], [30, 36].

We thus need to estimate the convolution product of two or more resurgent functions, both for concrete manipulations of resurgent functions in nonlinear contexts and for the foundations of the resurgence theory. For instance, such estimates will allow us to check that, when we come back to the resurgent series via $\mathscr{B}$, the exponential of a resurgent series is resurgent and that more generally one can substitute resurgent series in convergent power expansions, or define implicitly a resurgent series, or develop "alien calculus" when manipulating Écalle's alien derivatives. They will also show that the group of "formal tangent-to-identity diffeomorphisms at ∞", i.e., the group (for the composition law) $z + \mathbb{C}[[z^{-1}]]$, admits $z + \tilde{\mathscr{R}}$ as a subgroup, which is particularly useful for the study of holomorphic tangent-to-identity diffeomorphisms f (in this classical problem of local holomorphic dynamics [27], the Fatou coordinates have the same resurgent asymptotic expansion, the so-called direct iterator $f^* \in z + \tilde{\mathscr{R}}$ of [10]; thus its inverse, the inverse iterator, also belongs to $z+\tilde{\mathscr{R}}$, as well as its exponential, which appears in the Bridge equation connected with the "horn maps"—see § 3.3).

Such results of stability of the algebra of resurgent series under nonlinear operations are mentioned in Écalle's works, however the arguments there are sketchy and it was desirable to provide a proof.(1) Indeed, the subsequent authors dealing with resurgent series either took such results for granted or simply avoided resorting to them. The purpose of this article is to give clear statements with rigorous and complete proofs, so as to clarify the issue and contribute to make resurgence theory more accessible, hopefully opening the way for new applications of this powerful theory.

In this article, we shall deal with a particular case of resurgence called Ω-continuability or Ω-resurgence, which means that we fix in advance a discrete subset Ω of $\mathbb{C}$ and restrict ourselves to those resurgent functions whose analytic continuations have no singular point outside of Ω. Many interesting cases are already covered by this definition (one encounters Ω-continuable germs with $\Omega = \mathbb{Z}$ when dealing with differential equations formally conjugate to the Euler equation or in the study of the saddle-node singularities [11, 35], or with $\Omega = 2\pi i\mathbb{Z}$ when dealing with certain difference equations like Abel's equation for tangent-to-identity diffeomorphisms [10, 34], [8]). We preferred to restrict ourselves to this situation so as to make our method more transparent, even if more general definitions of resurgence can be handled—see Section 3.4. An outline of the article is as follows:

– In Section 2, we recall the precise definition of the corresponding algebras of resurgent functions, denoted by $\hat{\mathscr{R}}_\Omega$, and state Theorem 1, which is our main result on the control of the convolution product of an arbitrary number of Ω-continuable functions.

(1) This was one of the tasks undertaken in the seminal book [3] but, despite its merits, one cannot say that this book clearly settled this particular issue: the proof of the estimates for the convolution is obscure and certainly contains at least one mistake (see Remark 7.3).