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THE TROPICALIZATION OF THE MODULI SPACE OF CURVES

BY DAN ABRAMOVICH, LUCIA CAPORASO AND SAM PAYNE

Dedicated to Joe Harris.

ABSTRACT. – We show that the skeleton of the Deligne-Mumford-Knudsen moduli stack of stable curves is naturally identified with the moduli space of extended tropical curves, and that this is compatible with the "naive" set-theoretic tropicalization map. The proof passes through general structure results on the skeleton of a toroidal Deligne-Mumford stack. Furthermore, we construct tautological forgetful, clutching, and gluing maps between moduli spaces of extended tropical curves and show that they are compatible with the analogous tautological maps in the algebraic setting.

RÉSUMÉ. – On démontre que le squelette du champ des modules des courbes stables de Deligne-Mumford-Knudsen est naturellement identifié avec l'espace des modules des courbes tropicales de façon compatible avec l'application de tropicalisation « naïve » d'ensembles. La démonstration emploie des résultats généraux de structure sur le squelette des champs toroïdaux de Deligne-Mumford. En outre, on construit les morphismes tautologiques entre les espaces de modules des courbes tropicales étendues, et on démontre qu'ils sont compatibles avec leurs analogues dans le cadre algébrique.

1. Introduction

A number of researchers have introduced and studied the moduli spaces $M_{g,n}^{\mathrm{trop}}$, parametrizing certain metric weighted graphs called *tropical curves*, and exhibited analogies to the Deligne-Mumford-Knudsen moduli stacks of stable pointed curves, $\overline{\mathcal{M}}_{g,n}$, and to the Kontsevich moduli spaces of stable maps [38, 39, 23, 22, 35, 13, 36, 11, 12, 15, 16]. The paper [12] describes, in particular, an order reversing correspondence between the stratification of $M_{g,n}^{\mathrm{trop}}$ and the stratification of $\overline{\mathcal{M}}_{g,n}$, along with a natural compactification $\overline{M}_{g,n}^{\mathrm{trop}}$, the moduli space of *extended* tropical curves, where the correspondence persists. A seminal precursor for all of this work is the paper of Culler and Vogtmann on moduli of graphs and

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automorphisms of free groups, [18], in which a space of metric graphs called "outer space" was introduced.

The analogies between moduli of curves and moduli of graphs go further than the natural stratifications of compactifications. As we show in Section 8, the moduli spaces $\overline{M}_{g,n}^{\mathrm{trop}}$ admit natural maps

$$\pi_{g,n}^{\mathrm{trop}} : \overline{M}_{g,n+1}^{\mathrm{trop}} \to \overline{M}_{g,n}^{\mathrm{trop}}, \quad i = 1, \dots, n+1$$

associated to "forgetting the last marked point and stabilizing," analogous to the forgetful maps $\pi_{g,n}$ on the moduli spaces of curves. There are also clutching and gluing maps

$$\kappa_{g_1,n_1,g_2,n_2}^{\mathrm{trop}} : \overline{M}_{g_1,n_1+1}^{\mathrm{trop}} \times \overline{M}_{g_2,n_2+1}^{\mathrm{trop}} \to \overline{M}_{g_1+g_2,n_1+n_2}^{\mathrm{trop}}$$

and

$$\gamma_{g,n}^{\mathrm{trop}} : \overline{M}_{g-1,n+2}^{\mathrm{trop}} \to \overline{M}_{g,n}^{\mathrm{trop}}$$

covering the boundary strata of $\overline{M}_{g,n}^{\mathrm{trop}} \setminus M_{g,n}^{\mathrm{trop}}$, analogous to the corresponding clutching and gluing maps κ_{g_1,n_1,g_2,n_2} and $\gamma_{g,n}$ on the moduli spaces of curves. When the various subscripts g, n are evident we suppress them in the notation for these maps.

The main purpose of this paper is to develop these analogies into a rigorous and functorial correspondence. Write $\overline{M}_{g,n}$ for the coarse moduli space of $\overline{\mathcal{M}}_{g,n}$. We start with set-theoretic maps from the associated Berkovich analytic space $\overline{M}_{g,n}^{\mathrm{an}}$ to the tropical moduli space $\overline{M}_{g,n}^{\mathrm{trop}}$, described in Definition 1.1.1 below, and use Thuillier's construction of canonical skeletons of toroidal Berkovich spaces [47] to show that these maps are continuous, proper, surjective, and compatible with the tautological forgetful, clutching, and gluing maps. We work extensively with the combinatorial geometry of *generalized extended cone complexes*, as presented in Section 2.6.

To study the skeleton of $\overline{\mathcal{M}}_{g,n}$, we require a mild generalization of Thuillier's construction, presented in Section 6, below; the main technical results are Propositions 6.1.4, 6.1.8 and 6.2.6. Given a proper toroidal Deligne–Mumford stack $\mathscr{X}$ with coarse moduli space X, we functorially construct a generalized extended cone complex, the *skeleton* $\overline{\Sigma}(\mathscr{X})$, which is both a topological closed subspace of the Berkovich analytic space X^{an} associated to X, and also the image of a canonical retraction

$$\mathbf{p}_{\mathscr{X}} : X^{\mathrm{an}} \to \overline{\Sigma}(\mathscr{X}).$$

We emphasize that the skeleton $\overline{\Sigma}(\mathscr{X})$ depends on a toroidal structure on the stack $\mathscr{X}$, but lives in the analytification of the coarse moduli space X, which is not necessarily toroidal.

The compactified moduli space of tropical curves $\overline{M}_{g,n}^{\mathrm{trop}}$ is similarly a generalized extended cone complex, and one of our primary tasks is to identify the tropical moduli space $\overline{M}_{g,n}^{\mathrm{trop}}$ with the skeleton $\overline{\Sigma}(\overline{\mathcal{M}}_{g,n})$. See Theorem 1.2.1 for a precise statement.

1.1. The tropicalization map

There is a natural set theoretic *tropicalization map*

$$\mathrm{Trop} : \overline{M}_{g,n}^{\mathrm{an}} \to \overline{M}_{g,n}^{\mathrm{trop}},$$

well-known to experts [48, 5, 50], defined as follows. A point $[C]$ in $\overline{M}_{g,n}^{\mathrm{an}}$ is represented, possibly after a field extension, by a stable n-pointed curve C of genus g over the spectrum S

of a valuation ring R, with algebraically closed fraction field and valuation denoted val_C. Let $\mathbf{G}$ be the dual graph of the special fiber, as discussed in Section 3.2 below, where each vertex is weighted by the genus of the corresponding irreducible component, and with legs corresponding to the marked points. For each edge e_i in $\mathbf{G}$, choose an étale neighborhood of the corresponding node in which the curve is defined by a local equation $xy = f_i$, with f_i in R.

DEFINITION 1.1.1. – The tropicalization of the point $[C] \in \overline{M}_{g,n}^{\mathrm{an}}$ is the stable tropical curve $\mathbf{\Gamma} = (\mathbf{G}, \ell)$, with edge lengths given by

$$\ell(e_i) = \mathrm{val}_C(f_i).$$

See [50, Lemma 2.2.4] for a proof that the tropical curve $\mathbf{\Gamma}$ so defined is independent of the choices of R, C, étale neighborhood, and local defining equation, so the map Trop is well defined.

1.2. Main results

Our first main result identifies the map Trop with the projection from $\overline{M}_{g,n}^{\mathrm{an}}$ to its skeleton $\overline{\Sigma}(\overline{\mathcal{M}}_{g,n})$.

THEOREM 1.2.1. – *Let g and n be non-negative integers.*

1. *There is an isomorphism of generalized cone complexes with integral structure*

$$\Phi_{g,n} : \Sigma(\overline{\mathcal{M}}_{g,n}) \xrightarrow{\sim} M_{g,n}^{\mathrm{trop}}$$

extending uniquely to the compactifications

$$\overline{\Phi}_{g,n} : \overline{\Sigma}(\overline{\mathcal{M}}_{g,n}) \xrightarrow{\sim} \overline{M}_{g,n}^{\mathrm{trop}}.$$

2. *The following diagram is commutative:*

$$\begin{array}{ccc} \overline{M}_{g,n}^{\mathrm{an}} & \xrightarrow{\mathbf{p}_{\overline{\mathcal{M}}_{g,n}}} & \overline{\Sigma}(\overline{\mathcal{M}}_{g,n}) \\ & \searrow^{\mathrm{Trop}} & \downarrow{\scriptstyle \overline{\Phi}_{g,n}} \\ & & \overline{M}_{g,n}^{\mathrm{trop}}. \end{array}$$

In particular the map Trop *is continuous, proper, and surjective.*

The theorem is proven in Section 7.

Our second main result shows that the map Trop is compatible with the tautological forgetful, clutching, and gluing maps.

THEOREM 1.2.2. – *The following diagrams are commutative.*

The universal curve diagram:

$$\begin{array}{ccc} \overline{M}_{g,n+1}^{\mathrm{an}} & \xrightarrow{\mathrm{Trop}} & \overline{M}_{g,n+1}^{\mathrm{trop}} \\ {\scriptstyle \pi^{\mathrm{an}}}\downarrow & & \downarrow{\scriptstyle \pi^{\mathrm{trop}}} \\ \overline{M}_{g,n}^{\mathrm{an}} & \xrightarrow{\mathrm{Trop}} & \overline{M}_{g,n}^{\mathrm{trop}}, \end{array}$$