ISSN 0012-9593

quatrième série - tome 48 fascicule 4 juillet-août 2015

ANNALES
SCIENTIFIQUES
de
L'ÉCOLE
NORMALE
SUPÉRIEURE

Marius JUNGE & Carlos PALAZUELOS & Javier PARCET &
Mathilde PERRIN & Éric RICARD

# *Hypercontractivity for free products*

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

# Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique



**Rédaction / *Editor***

Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
`annales@ens.fr`

**Édition / *Publication***

Société Mathématique de France
Institut Henri Poincaré
11, rue Pierre et Marie Curie
75231 Paris Cedex 05
Tél. : (33) 01 44 27 67 99
Fax : (33) 01 40 46 90 96

**Abonnements / *Subscriptions***

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 09
Fax : (33) 04 91 41 17 51
email : `smf@smf.univ-mrs.fr`

**Tarifs**

Europe : 515 €. Hors Europe : 545 €. Vente au numéro : 77 €.



ISSN 0012-9593

Directeur de la publication : Marc Peigné
Périodicité : 6 nos / an

*Ann. Scient. Éc. Norm. Sup.*
4$^e$ série, t. 48, 2015, p. 861 à 889

# HYPERCONTRACTIVITY FOR FREE PRODUCTS

BY MARIUS JUNGE, CARLOS PALAZUELOS, JAVIER PARCET, MATHILDE PERRIN AND ÉRIC RICARD

ABSTRACT. – In this paper, we obtain optimal time hypercontractivity bounds for the free product extension of the Ornstein-Uhlenbeck semigroup acting on the Clifford algebra. Our approach is based on a central limit theorem for free products of spin matrix algebras with mixed commutation/anticommutation relations. With another use of Speicher's central limit theorem, we can also obtain the same bounds for free products of $q$-deformed von Neumann algebras interpolating between the fermonic and bosonic frameworks. This generalizes the work of Nelson, Gross, Carlen/Lieb and Biane. Our main application yields hypercontractivity bounds for the free Poisson semigroup acting on the group algebra of the free group $\mathbb{F}_n$, uniformly in the number of generators.

RÉSUMÉ. – Cet article s'intéresse à des estimations hypercontractives pour des semi-groupes obtenus comme produits libres. Notre approche est basée sur un théorème de la limite centrale pour des produits libres d'algèbres de spin ou autres. Nous obtenons un temps optimal d'hypercontractivité $L_p \to L_q$ pour les produits libres des semi-groupes d'Orstein-Uhlenbeck sur les algèbres $q$-déformées $(-1 \leq q \leq 1)$ qui interpolent entre les fermions $(q = -1)$ et les bosons $(q = 1)$. Ces résultats s'inspirent des travaux de Nelson, Gross, Carlen/Lieb et Biane et les généralisent. Comme application, nous déduisons un temps d'hypercontractivité $L_p \to L_q$ pour le semi-groupe de Poisson libre sur l'algèbre du groupe libre à une infinité de générateurs.

## Introduction

Hypercontractivity is a way to quantify the regularizing effect of certain well behaved semigroups in terms of $L_p$ integrability. More precisely, let $(\Omega, \Sigma, \mu)$ be a probability space and consider a Markov semigroup of operators $(\mathcal{S}_t)_{t\geq 0}$ acting on $\Sigma$-measurable functions. This roughly means that $\mathcal{S}_t$ is self-adjoint on $L_2(\Omega)$ and preserves constant functions and positivity. The Ornstein-Uhlenbeck process on $\mathbb{R}^n$ equipped with the Gaussian measure is a model example. The hypercontractivity problem for $1 < p \leq q < \infty$ consists in determining the optimal time $t_{p,q} > 0$ above which the following inequality holds

$$\Big( \int_\Omega \big| \mathcal{S}_t f(\omega) \big|^q \, d\mu(\omega) \Big)^{\frac{1}{q}} \leq \Big( \int_\Omega \big| f(\omega) \big|^p \, d\mu(\omega) \Big)^{\frac{1}{p}} \quad \text{for all} \quad t \geq t_{p,q}.$$

The existence of such value $t_{p,q}$ is suggested by elementary semigroup theory. Namely, given a Markov semigroup as above one can find nonnegative numbers $(\psi(k))_{k \geq 1}$ and eigenfunctions $(f_k)_{k \geq 1}$ so that $\mathcal{S}_t f_k = e^{-t\psi(k)} f_k$. Given $f$ in the span of the $f_k$'s, this shows why $\mathcal{S}_t f$ gains integrability for $t$ large.

The phenomenon of hypercontractivity was discovered independently and almost simultaneously in harmonic analysis and quantum field theory. In the context of harmonic analysis, Bonami [6] introduced hypercontractivity for classical Poisson semigroups motivated by the relation between the integrability of a function and the decay properties of its Fourier coefficients. On the other hand, Nelson [30] considered hypercontractivity for classical Ornstein-Uhlenbeck semigroups to bound from below certain Hamiltonians arising in quantum field theory. In the first case the eigenfunctions are given by the trigonometric system, and in the second by Gaussian chaos, see below for further details. The work of Gross [15] establishes an intimate connection between hypercontractivity and the logarithmic Sobolev inequalities, a limiting dimension-free form of Sobolev embedding.

The starting point in this subject is the so-called two-point inequality, which was first proved by Bonami and rediscovered years later by Gross [6, 15]. This inequality was also instrumental in Beckner's theorem on the optimal constants for the Hausdorff-Young inequality [2] and Gross used it as a key step towards his logarithmic Sobolev inequalities. More recently, the two-point inequality has also produced very important applications in computer science and in both classical and quantum information theory [8, 11, 21, 22]. If $1 < p \leq q < \infty$ and $\alpha, \beta \in \mathbb{C}$, Bonami-Gross inequality can be written as follows

$$\Big( \sum_{\varepsilon = \pm 1} \Big| \frac{(1 + \varepsilon e^{-t})\alpha + (1 - \varepsilon e^{-t})\beta}{2^{1 + \frac{1}{q}}} \Big|^q \Big)^{\frac{1}{q}} \leq \Big( \frac{|\alpha|^p + |\beta|^p}{2} \Big)^{\frac{1}{p}} \ \Leftrightarrow \ t \geq \frac{1}{2} \log \frac{q-1}{p-1}.$$

Under Bonami's viewpoint, this inequality means that the "Poisson semigroup" on the group $\mathbb{Z}_2$ is hypercontractive with optimal constant. Gross understood it as the optimal hypercontractivity bound for the Ornstein-Uhlenbeck semigroup on the Clifford algebra with one generator $\mathscr{C}(\mathbb{R})$. Although the two-point inequality can be generalized in both directions, harmonic analysis has evolved towards other related norm inequalities in the classical groups—like $\Lambda_p$ sets in $\mathbb{Z}$—instead of analyzing hypercontractivity over the compact dual of other discrete groups. Namely, to the best of our knowledge only hypercontractivity for the Cartesian products of $\mathbb{Z}_2$ and $\mathbb{Z}$ has been understood so far, see [42]. Motivated by the recent development of noncommutative analysis and free probabilities, the first goal of this paper is to replace Cartesian products by free products, and thereby obtain hypercontractivity inequalities for the free Poisson semigroups acting on the group von Neumann algebras associated to $\mathbb{F}_n = \mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}$ and $\mathbb{G}_n = \mathbb{Z}_2 * \mathbb{Z}_2 * \cdots * \mathbb{Z}_2$.

Let $\mathrm{G}$ denote any of the free products considered above and let $\lambda : \mathrm{G} \to \mathcal{B}(\ell_2(\mathrm{G}))$ stand for the corresponding left regular representation. The group von Neumann algebra $\mathscr{L}(\mathrm{G})$ is the weak operator closure of the linear span of $\lambda(\mathrm{G})$. If $e$ denotes the identity element of G, the algebra $\mathcal{L}(\mathrm{G})$ comes equipped with the standard trace $\tau(f) = \langle \delta_e, f\delta_e \rangle$. Let $L_p(\mathcal{L}(\mathrm{G}), \tau)$ be the $L_p$ space over the noncommutative measure space $(\mathcal{L}(\mathrm{G}), \tau_{\mathrm{G}})$—the so called noncommutative $L_p$ space—with norm $\|f\|_p^p = \tau|f|^p$. We invite the reader to check that $L_p(\mathcal{L}(\mathrm{G}), \tau) = L_p(\mathbb{T})$ for $\mathrm{G} = \mathbb{Z}$ after identifying $\lambda_{\mathbb{Z}}(k)$ with $e^{2\pi i k \cdot}$. In the general case, the absolute value and the power $p$ are obtained from functional calculus for this (unbounded) operator on the Hilbert space $\ell_2(\mathrm{G})$, see [35] for details. If $f = \sum_g \widehat{f}(g)\lambda(g)$, the free Poisson semigroup on G is given by the family of linear maps

$$\mathcal{P}_{\mathrm{G},t} f = \sum_{g \in \mathrm{G}} e^{-t|g|} \widehat{f}(g)\lambda(g) \quad \text{with} \quad t \in \mathbb{R}_+.$$

In both cases $\mathrm{G} \in \{\mathbb{F}_n, \mathbb{G}_n\}$, $|g|$ refers to the Cayley graph length. In other words, $|g|$ is the number of letters (generators and their inverses) which appear in $g$ when it is written in reduced form. It is known from [17] that $\mathcal{P}_{\mathrm{G}} = (\mathcal{P}_{\mathrm{G},t})_{t \geq 0}$ defines a Markov semigroup on $\mathcal{L}(\mathrm{G})$. In particular, $\mathcal{P}_{\mathrm{G},t}$ defines a contraction on $L_p(\mathcal{L}(\mathrm{G}))$ for every $1 \leq p \leq \infty$. In our first result we provide new hypercontractivity bounds for the free Poisson semigroups on those group von Neumann algebras. If $g_1, g_2, \ldots, g_n$ stand for the free generators of $\mathbb{F}_n$, we will also consider the symmetric subalgebra $\mathcal{A}_{\mathrm{sym}}^n$ of $\mathcal{L}(\mathbb{F}_n)$ generated by the self-adjoint operators $\lambda(g_j) + \lambda(g_j)^*$. In other words, we set

$$\mathcal{A}_{\mathrm{sym}}^n = \big\langle \lambda(g_1) + \lambda(g_1)^*, \ldots, \lambda(g_n) + \lambda(g_n)^* \big\rangle''.$$

THEOREM A. – *If $1 < p \leq q < \infty$, we find*:

i) *Optimal time hypercontractivity for $\mathbb{G}_n$*

$$\big\| \mathcal{P}_{\mathbb{G}_n,t} : L_p(\mathcal{L}(\mathbb{G}_n)) \to L_q(\mathcal{L}(\mathbb{G}_n)) \big\| = 1 \ \Leftrightarrow \ t \geq \frac{1}{2} \log \frac{q-1}{p-1}.$$

ii) *Hypercontractivity for $\mathbb{F}_n$ over twice the optimal time*

$$\big\| \mathcal{P}_{\mathbb{F}_n,t} : L_p(\mathcal{L}(\mathbb{F}_n)) \to L_q(\mathcal{L}(\mathbb{F}_n)) \big\| = 1 \quad \textit{if} \quad t \geq \log \frac{q-1}{p-1}.$$

iii) *Optimal time hypercontractivity in the symmetric algebra $\mathcal{A}_{\mathrm{sym}}^n$*

$$\big\| \mathcal{P}_{\mathbb{F}_n,t} : L_p(\mathcal{A}_{\mathrm{sym}}^n) \to L_q(\mathcal{A}_{\mathrm{sym}}^n) \big\| = 1 \ \Leftrightarrow \ t \geq \frac{1}{2} \log \frac{q-1}{p-1}.$$

Theorem A i) extends Bonami's theorem for $\mathbb{Z}_2^n$ to the free product case with optimal time estimates. According to the applications in complexity theory and quantum information of Bonami's result, it is conceivable that Theorem A could be of independent interest in those areas. These potential applications will be explored in further research. Theorem A ii) gives the first hypercontractivity estimate for the free Poisson semigroup on $\mathbb{F}_n$, where a factor 2 is lost from the expected optimal time. This is related to our probabilistic approach to the problem and a little distortion must be done to make $\mathbb{F}_n$ fit in. Theorem A iii) refines this, providing optimal time estimates in the symmetric algebra $\mathcal{A}_{\mathrm{sym}}^n$. We also obtain optimal time $L_p \to L_2$ hypercontractive estimates for linear combinations of words with length less than or equal to one. Apparently, our probabilistic approach in this paper is limited to go beyond the constant 2 in the general case. We managed to push it to $1 + \frac{1}{4} \log 2 \sim 1.173$ in the last section. Actually, we have recently found in [19] an alternative combinatorial/numerical method which yields optimal $L_2 \to L_q$ estimates for $q \in 2\mathbb{Z}$ for $\mathbb{F}_2$ and other groups, and